43. STIELTJES INTEGRAL

The Riemann integral admits a natural generalization that was discovered by T.J. Stieltjes in the 1890s in his work on continued fractions. This generalization is quite useful in mathematics itself as well as applications and also serves as a foundation for the corresponding extension of the Lebesgue integration theory. (Stieljes had an interesting career path to mathematics. He died in 1894 at the age of 38, too early to see his work published and his integral gain prominence.)

43.1 Stieltjes integral: definition and motivation.

The main distinction of the Stietjes integral from the Riemann integral is that it depends on two functions: First, the integrand f and then a function g that replaces the identity map (i.e., $g(x) = x$) in the increment of the underlying variable. Precise definitions are as follows:

Definition 43.1 (Stieltjes integral) *Let $a < b$ be reals and $f, g\colon [a,b] \to \mathbb{R}$ functions. Given a marked partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ — i.e., two sequences of reals subject to the requirements (36.1–36.2) — we define the* Riemann-Stieltjes sum *by*

$$S(f, \mathrm{d}g, \Pi) := \sum_{i=1}^n f(t_i^\star)\big(g(t_i) - g(t_{i-1})\big) \tag{43.1}$$

The function f is then said to be Stieltjes integrable with respect to g on $[a,b]$ in Riemann sense *(or sometimes called* Riemann-Stieltjes integrable*) if*

$$\int_a^b f\mathrm{d}g := \lim_{\|\Pi\|\to 0} S(f, \mathrm{d}g, \Pi) \text{ exists} \tag{43.2}$$

where the "limit" abbreviates the same concept as for the Riemann integral (see Definition 36.2). We call the object on the left the Stieltjes integral of f with respect to g.

Some remarks are in order. The above is often referred to as the *Riemann-Stieltjes integral*. This is not because Riemann had anything to do with it but rather that the integral uses the framework of the Riemann integral. (A version called *Lebesgue-Stieltjes integral* exists in Lebesgue theory of integration.) The Riemann integral is thus a special case of the (Riemann-)Stieltjes integral; indeed,

$$\big(\forall x \in [a,b]\colon\ g(x) = x\big) \ \Rightarrow\ \int_a^b f\mathrm{d}g = \int_a^b f(x)\mathrm{d}x \tag{43.3}$$

Turning this around, the Stieltjes integral allows us to generalize the notion of area to the situation when the "length" of interval $[s,t]$ is given $g(t) - g(s)$ and area of the rectangle $[s,t] \times [0,h]$ is thus $h[g(t) - g(s)]$. Notably, this includes negative "lengths" but this is no problem because the Riemann integral anyway computes the signed area.

The Stieltjes integral naturally arises and is generally quite useful in applied disciplines. Indeed, starting in physics, the work exerted by time t by a force $F = F(t,x)$ acting on a particle moving along a trajectory described by $x = x(t)$ is written by physicists as $\int_0^t F(s, x(s))\mathrm{d}x(s)$, which must be interpreted via Stieltjes integral.

Another natural application of the Stieltjes integral is in differential geometry, where we may want to "integrate f along a curve given by the graph of g." A particular case where this shows us is complex analysis where we want to give meanings to the integral

$$\int_\Gamma f(z)\mathrm{d}z \tag{43.4}$$

where z and $f(z)$ are generally complex valued and Γ represents the set of points swept by a complex valued function $t \mapsto \gamma(t)$ for t ranging over a given interval.

Another useful application of the Stieltjes integral arises in in probability, where g represents the *cumulative distribution function* of a random variable X which means that

$$g(x) := P(X \leqslant x) \tag{43.5}$$

The integral $\int_a^b f\mathrm{d}g$ then corresponds to the *expectation* of $f(X)$; i.e., the statistical "mean value" of the random variable obtained by plugging X into f. The use of the Stieltjes integral permits treating all the various kinds of distributions of X — namely, discrete, continuous and mixtures thereof — under the same umbrella.

The fact that g need not be monotone, and the increments $g(t) - g(s)$ over interval $[s, t]$ thus need not be positive, is quite advantageous in applications. For instance, consider a position in stock portfolio whose volume at time t is described by $g(t)$. The quantity

$$f(t_i^\star)\bigl[g(t_i) - g(t_{i-1})\bigr] \tag{43.6}$$

is then the gain (if positive) or loss (if negative) of the value of the portfolio over time interval $[t_{i-1}, t_i]$ assuming all trade was executed at the price at time $t_i^\star \in [t_{i-1}, t_i]$. (One can also think about these the other way, with f representing the position and g the value.) The Stietljes sum thus approximates the total cash value traded over the time interval $[a, b]$ which, in the limit as the as mesh of the partition tends to zero, is thus given by the integral $\int_a^b f\mathrm{d}g$.

We note that Rudin's book presents a different definition of the Stieltjes integral which is based on Darboux's approach to Riemann integration. This streamlines the analysis somewhat but forces us to work with g monotone or, relying on the Jordan decomposition, of bounded variation. The applications mentioned earlier are not always of this kind and so we prefer to work with the Stieltjes integral in the Riemann sense.

43.2 Properties of Stieltjes integral.

We now move to discuss the basic properties of Stieltjes integral. Let us henceforth use

$$\mathrm{RS}\bigl(h, [a, b]\bigr) := \Bigl\{f\colon [a, b] \to \mathbb{R} :\ \int_a^b f\mathrm{d}h \text{ exists}\Bigr\} \tag{43.7}$$

to denote the set of functions that are Stieltjes-integrable with respect to g on $[a, b]$ in Riemann sense. We start with some "good news;" namely, facts where the Stieltjes integral behaves very much as Riemann's:

Lemma 43.2 (Linearity) *Let $a < b$ be reals and let $h\colon [a, b] \to \mathbb{R}$ be given. Then for all $f, g \in \mathrm{RS}(h, [a, b])$ and all $\alpha, \beta \in \mathbb{R}$,*

$$\alpha f + \beta g \in \mathrm{RS}\bigl(h, [a, b]\bigr) \tag{43.8}$$

and

$$\int_a^b \bigl(\alpha f + \beta g\bigr)\mathrm{d}h = \alpha \int_a^b f\mathrm{d}h + \beta \int_a^b g\mathrm{d}h \tag{43.9}$$

Proof. The proof is the exactly the same as for the Riemann integral (see Lemma 36.5) and so we leave it to the reader. ☐

A similar type of linearity holds also for the integrator:

Lemma 43.3 *For all $\alpha, \beta \in \mathbb{R}$, all $g_1, g_2 \colon [a,b] \to \mathbb{R}$ and all $f \in \mathrm{RS}(g_1,[a,b]) \cap \mathrm{RS}(g_2,[a,b])$,*

$$f \in \mathrm{RS}\big(\alpha g_1 + \beta g_2, [a,b]\big) \;\wedge\; \int_a^b f \mathrm{d}(\alpha g_1 + \beta g_2) = \alpha \int_a^b f \mathrm{d} g_1 + \beta \int_a^b f \mathrm{d} g_2 \tag{43.10}$$

Proof. Left to the reader. ☐

Moving to properties where the Stietljes integral behaves somewhat differently than (its special case of) the Riemann integral we note that the boundedness that came with Riemann integrability is no longer applicable. Indeed, on intervals $[s,t] \subseteq [a,b]$ where g is constant and the values of $\{f(x) \colon x \in (s,t)\}$ do not contribute to $S(f, \mathrm{d}g, \Pi)$ for any partition Π and are thus completely unconstrained by the assumption that f is integrable with respect to g. Turning to the positive side of the story, we get:

Lemma 43.4 *Let $f, g \colon [a,b] \to \mathbb{R}$ be such that $f \in \mathrm{RS}(g,[a,b])$. Then there exists a partition $\Pi = \{t_i\}_{i=0}^n$ of $[a,b]$ such that*

$$\forall i = 1, \dots, n \colon \quad \sup_{x \in [t_{i-1}, t_i]} \big|f(x)\big| < \infty \;\vee\; \big(\forall x, y \in [t_{i-1}, t_i] \colon\; g(x) = g(y)\big) \tag{43.11}$$

In particular, if g is NOT constant on any non-degenerate closed subinterval of $[a,b]$, then $f \in \mathrm{RS}(g,[a,b])$ implies that f is bounded.

As it turns out, also the additivity property with respect to the underlying domain holds only in a restricted sense. Namely, we have:

Lemma 43.5 *For all reals $a < c < b$ and all $f, g \colon [a,b] \to \mathbb{R}$,*

$$f \in \mathrm{RS}(g,[a,b]) \;\Rightarrow\; f \in \mathrm{RS}(g,[a,c]) \wedge f \in \mathrm{RS}(g,[c,b]) \tag{43.12}$$

and, in particular,

$$\forall f \in \mathrm{RS}(g,[a,b]) \colon \quad \int_a^b f \mathrm{d}g = \int_a^c f \mathrm{d}g + \int_c^b f \mathrm{d}g \tag{43.13}$$

Proof. Left to the reader. ☐

The previous lemma worked because it assumed integrability on the larger domain. Unlike the Riemann integral, for which (43.12) is an equivalence, for the Stieltjes integral one can have integrability on $[a,c]$ and on $[c,b]$ without having integrability on $[a,b]$. This is because:

Lemma 43.6 (Necessary condition for RS-integrability) *Let $f, g \colon [a,b] \to \mathbb{R}$ be such that $f \in \mathrm{RS}(g,[a,b])$. Then for each $\epsilon > 0$ there is $\delta > 0$ such that for any (unmarked) partition $\Pi = (\{t_i\}_{i=0}^n)$ of $[a,b]$,*

$$\|\Pi\| < \delta \;\Rightarrow\; \sum_{i=1}^n \mathrm{osc}\big(f, [t_{i-1}, t_i]\big) \big|g(t_i) - g(t_{i-1})\big| < \epsilon \tag{43.14}$$

In particular, if $f \in \mathrm{RS}(g,[a,b])$, then f and g have no common discontinuity points.

Proof. To get (43.14), consider two Stieltjes sums for partitions $\Pi^\star$ and $\Pi^{\star\star}$ with the same partition points $\{t_i\}_{i=0}^n$ but the marked points $t_i^\star, t_i^{\star\star} \in [t_{i-1}, t_i]$ chosen such that, for each $i = 1, \dots, n$, the difference $f(t_i^\star) - f(t_i^{\star\star})$ has the same sign as $g(t_i) - g(t_{i-1})$ and has absolute value at least $\frac{1}{2}\text{osc}(f, [t_{i-1}, t_i])$. Then

$$\frac{1}{2}\sum_{i=1}^n \text{osc}\bigl(f, [t_{i-1}, t_i]\bigr)\bigl|g(t_i) - g(t_{i-1})\bigr| \leqslant S(f, \mathrm{d}g, \Pi^\star) - S(f, \mathrm{d}g, \Pi^{\star\star}) \tag{43.15}$$

Assuming Stieltjes integrability $f \in \text{RS}(g, [a, b])$ and δ related to δ as in Definition 36.2, the right-hand sid aller than 2ϵ once $\|\Pi^\star\| = \|\Pi^{\star\star}\| < \delta$.

For the second part note that suppose first that f fails to be right-continuous at some $x \in [a, b)$. Using the same ideas as in Lemma 39.5(2), this implies

$$\eta := \inf_{\delta \in (0, b-x)} \text{osc}(f, [x, x+\delta]) > 0 \tag{43.16}$$

Letting $\delta > 0$ be related to $\epsilon > 0$ as in (43.14), for partitions where x is among the partition points we then get

$$\forall t \in (x, x+\delta) \cap [a, b]\colon \quad \bigl|g(t) - g(x)\bigr| < \frac{1}{\eta}\epsilon \tag{43.17}$$

As this holds for all $\epsilon > 0$, we conclude that g is right-continuous at x. A similar proof gives left-continuity of g at all points $x \in (a, b]$ where f is NOT left continuous.

It remains to rule out the possibility that, at some $x \in (a, b)$, the function f is NOT right-continuous and g is NOT left-continuous but is right continuous. Here we take a partition that does NOT contain x among its partition points. Then we still have

$$\tilde{\eta} := \inf_{a \leqslant s < x < t \leqslant b} \text{osc}\bigl(f, [s, t]\bigr) > 0 \tag{43.18}$$

and so (43.14) gives (for δ related to ϵ as above),

$$\forall s, t \in [a, b]\colon\ s < x < t \ \wedge\ t - s < \delta \ \Rightarrow\ \bigl|g(t) - g(s)\bigr| < \frac{1}{\tilde{\eta}}\epsilon \tag{43.19}$$

If g is assumed right continuous, then taking $t \to x^+$ shows that $|g(x) - g(s)| \leqslant \tilde{c}^{-1}\epsilon$ whenever $0 < x - s < \delta$. But then g is left-continuous as well. (The complementary set of type of dicontinuities is handled similarly.) □

The previous proof now explains why $\Leftarrow$ generally fails in (43.12): We may have $f \in \text{RS}(g, [a, c]) \cap \text{RS}(g, [c, b])$ but with f right-continuous but NOT left-continuous at c and g left-continuous but NOT right-continuous at c. Then f and g have a common discontinuity point at c and so $f \notin \text{RS}(g, [a, b])$.

What we just showed is actually a somewhat annoying feature of the Stieltjes integral which stems from the fact that, for g with jumps, we may not be able to refine partition intervals so that the increment over each of them is small. This is partially fixed in:

Definition 43.7 (Generalized Stieltjes integrability) *We say that* $f\colon [a, b] \to \mathbb{R}$ *is* generalized Stieltjes integrable with respect to $g\colon [a, b] \to \mathbb{R}$ *if there is* $L \in \mathbb{R}$ *for each* $\epsilon > 0$ *there is* $\delta > 0$ *and an (unmarked) partition* Π_ϵ *such that for all partitions* Π

$$\Pi_\epsilon \subseteq \Pi \ \wedge\ \|\Pi\| < \delta \ \Rightarrow\ \bigl|S(f, \mathrm{d}g, \Pi) - L\bigr| < \epsilon \tag{43.20}$$

Here $\Pi_\epsilon \subseteq \Pi$ *means that the partition points of* Π *include all the partition points of* Π_ϵ*.*

Note that we can restrict Π_ϵ and δ to $\|\Pi_\epsilon\| < \delta$, which then forces $\|\Pi\| < \delta$ as soon as $\Pi_\epsilon \subseteq \Pi$. The reference to δ is thus redundant and we can leave it out altogether. The resulting integral is called the *Moore-Pollard-Stieltjes integral* by some authors. While, with these generalizations, the integral becomes additive in the underlying domain, this is not such an important improvement and so we will stick to the definition used previously because it makes the proofs somewhat easier.

43.3 Integration by parts and substitution rule.

The Stieltjes integral retains (or extends) various properties valid for the Riemann integral; namely, the Integration by parts from Corollary 40.8 and the Substitution rule from Corollary 40.9. The proof of these for the Riemann integral relied on the Fundamental Theorem of Calculus which for the Stieltjes integral takes the form:

Lemma 43.8 (Reduction to Riemann integral) *Let $f, g\colon [a,b] \to \mathbb{R}$ be such that*

(1) *f is Riemann integrable on $[a,b]$, and*
(2) *g is continuous on $[a,b]$, differentiable on (a,b) with g' Riemann integrable on $[a,b]$.*

Then f is Stieltjes integrable with respect to g on $[a,b]$ and

$$\int_a^b f \mathrm{d}g = \int_a^b f(x)g'(x)\mathrm{d}x \tag{43.21}$$

Proof. The proof is very similar to the proof of Theorem 40.7 and so we omit it. □

Note that Lemma 43.8 subsumes the FTCII thanks to the fact that $\int_a^b 1\mathrm{d}g = g(b) - g(a)$, as follows from $S(1, \mathrm{d}g, \Pi) = g(b) - g(a)$ for any partition Π. While the reduction to the Riemann integral is often used to evaluate the Stieltjes integral, the Integration by Parts and Substitution rule we will state and prove next do NOT rely on this reduction and are thus more general. Indeed, we have:

Lemma 43.9 (Integration by parts) *For all $f, g\colon [a,b] \to \mathbb{R}$*

$$f \in \mathrm{RS}(g, [a,b]) \;\Leftrightarrow\; g \in \mathrm{RS}(f, [a,b]) \tag{43.22}$$

and, if both TRUE, then

$$\int_a^b f \mathrm{d}g = f(b)g(b) - f(a)g(a) - \int_a^b g\, \mathrm{d}f \tag{43.23}$$

Proof. The key point of the proof is that, for any marked partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ of $[a,b]$, the pair $\Pi' = (\{t_i^\star\}_{i=0}^{n+1}, \{t_{i-1}\}_{i=1}^{n+1})$, where $t_0^\star := a$ and $t_{n+1}^\star := b$, is another marked partition of $[a,b]$. A calculation shows

$$\begin{aligned} f(a)g(a) + S(f, \mathrm{d}g, \Pi) &= f(a)g(a) + \sum_{i=1}^n f(t_i^\star)\big(g(t_i) - g(t_{i-1})\big) \\ &= \sum_{i=0}^n f(t_i^\star)g(t_i) - \sum_{i=0}^n f(t_{i+1}^\star)g(t_i) + f(b)g(b) \\ &= \sum_{i=0}^n g(t_i)\big[f(t_i^\star) - f(t_{i+1}^\star)\big] + f(b)g(b) = -S(g, \mathrm{d}f, \Pi') + f(b)g(b) \end{aligned} \tag{43.24}$$

Now assume that $g \in \mathrm{RS}(f,[a,b])$ and pick $\epsilon > 0$. Then $|S(g,\mathrm{d}f,\Pi') - \int_a^b g\mathrm{d}f| < \epsilon$ as soon as $\|\Pi'\| < \delta$, for δ related to ϵ as in Definition 36.2. But $\|\Pi'\| \leqslant 2\|\Pi\|$ and so if $\|\Pi\| < \delta/2$, then (43.24) shows that $S(f,\mathrm{d}g,\Pi)$ is within ϵ of the right-hand side of (43.23). It follows that $f \in \mathrm{RS}(g,[a,b])$, proving $\Leftarrow$ in (43.22), and that the identity (43.23) holds. (The equivalence in (43.22) holds by symmetry.) $\square$

Note that writing $\mathrm{d}g = g'(x)\mathrm{d}x$ and $\mathrm{d}f = f'(x)\mathrm{d}f$, the previous lemma subsumes the statement of Corollary 40.8. Note that combining Lemmas 43.9 and 43.9 we get:

Corollary 43.10 *Suppose $f,g\colon [a,b] \to \infty$ are such that f is Riemann integrable and g is continuous on $[a,b]$. Moreover, assume that g is differentiable on $[a,b]$ with g' Riemann integrable. Then $g \in \mathrm{RS}(f,[a,b])$ and*

$$\int_a^b g\,\mathrm{d}f = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\mathrm{d}x \tag{43.25}$$

Note that this again amounts to the use of the formal expression $\mathrm{d}g = g'(x)\mathrm{d}x$. Moving to the Substitution rule, here we get:

Lemma 43.11 (Substitution) *Let $g,h\colon [a,b] \to \mathbb{R}$ and, assuming $g \in \mathrm{RS}(h,[a,b])$, let*

$$\forall x \in [a,b]\colon\ G(x) := \int_a^x g\,\mathrm{d}h \tag{43.26}$$

Then for all bounded $f\colon [a,b] \to \mathbb{R}$,

$$f \in \mathrm{RS}(G,[a,b]) \ \Leftrightarrow\ f\cdot g \in \mathrm{RS}(h,[a,b]) \tag{43.27}$$

and, assuming that both sides are TRUE,

$$\int_a^b f\mathrm{d}G = \int_a^b f\cdot g\,\mathrm{d}h \tag{43.28}$$

Here $(f\cdot g)(x) := f(x)g(x)$.

Proof. Assume $g \in \mathrm{RS}(h,[a,b])$. The key point of the proof is the following approximation claim: For each $\epsilon > 0$ there is $\delta > 0$ for any marked partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ with $\|\Pi\| < \delta$, we have

$$\sum_{i=1}^n \left| g(t_i^\star)\big[h(t_i) - h(t_{i-1})\big] - \int_{t_{i-1}}^{t_i} g\mathrm{d}h \right| < \epsilon \tag{43.29}$$

To see why this is true, let $\delta > 0$ be related to $\epsilon > 0$ as in the definition of Stieltjes integrability. Given a partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ with $\|\Pi\| < \delta$, let

$$I := \left\{ i = 1,\dots,n\colon \int_{t_{i-1}}^{t_i} g\mathrm{d}h > g(t_i^\star)\big[h(t_i) - h(t_{i-1})\big] \right\} \tag{43.30}$$

Since $g \in \mathrm{RS}(h,[t_{i-1},t_i])$, for each $i \in I$ there is a partition Π_i of $[t_{i-1},t_i]$ such that

$$\left| S(g,\mathrm{d}h,\Pi_i) - \int_{t_{i-1}}^{t_i} g\mathrm{d}h \right| < \frac{\epsilon}{n} \tag{43.31}$$

Now consider the partition $\widetilde{\Pi}$ that contains all partition points of the partitions Π and Π_i for all $i \in I$, and the marked points of Π in intervals indexed by $i \notin I$ and all the marked points of the partitions Π_i with $i \in I$. The additivity of the Stieltjes sum and the integral then shows

$$S(f,\mathrm{d}g,\widetilde{\Pi}) - \int_a^b g\mathrm{d}h = \sum_{i=1}^n \biggl(g(t_i^\star)\bigl[h(t_i)-h(t_{i-1})\bigr] - \int_{t_{i-1}}^{t_i} g\mathrm{d}h \biggr)^+ + \sum_{i\in I}\Bigl(S(g,\mathrm{d}h,\Pi_i) - \int_{t_{i-1}}^{t_i} g\mathrm{d}h\Bigr) \tag{43.32}$$

where the use of the positive part $a^+ := \max\{0,a\}$ effectively eliminates terms with $i \in I$ from the first sum. Since $\|\widetilde{\Pi}\| < \delta$, the left-hand side is at least $-\epsilon$. The fact that (43.31) holds for each Π_i with $i \in I$ in turn ensures that the second sum on the right is at most ϵ. Hence we get

$$\sum_{i=1}^n \biggl(g(t_i^\star)\bigl[h(t_i)-h(t_{i-1})\bigr] - \int_{t_{i-1}}^{t_i} g\mathrm{d}h \biggr)^+ < 2\epsilon \tag{43.33}$$

Since the same applies to the sum of the negative parts, we get (43.29) with 4ϵ instead of ϵ on the right-hand side.

Using (43.29) we now quickly finish the claim. Let $f\colon [a,b] \to \mathbb{R}$ and let Π be a partition of $[a,b]$. Then (43.26) and additivity of the integral give

$$S(f\cdot g,\mathrm{d}h,\Pi) - S(f,\mathrm{d}G,\Pi) = \sum_{i=1}^n f(t_i^\star)\biggl[g(t_i^\star)\bigl[h(t_i)-h(t_{i-1})\bigr] - \int_{t_{i-1}}^{t_i} g\mathrm{d}h \biggr] \tag{43.34}$$

Assuming that f is bounded, the right-hand side is bounded by $\|f\|$ times the quantity in (43.29). So the convergence of $S(f,\mathrm{d}g,\Pi)$ as $\|\Pi\| \to 0$ is equivalent to the convergence of $S(f,\cdot g,\mathrm{d}h,\Pi)$ and both "limits" (if they exist) are equal. □

As it turns out, the conclusion of Lemma 43.11 may fail when f is unbounded. Indeed, set $a := 0$, $b := 1$, $g(x) = h(x) := x$ and $f(x) := 1/x$ for $x > 0$ and $f(0) = 0$. Then $G(x) = \frac{1}{2}x^2$ and $f \notin \mathrm{RS}(G,[0,1])$ because f is unbounded on intervals on which G is non-constant. Yet $f\cdot g = 1$ and so $f\cdot g \in \mathrm{RS}(h,[0,1])$. This is clearly because the Stieltjes integral works only with finite partitions which does not allow refinements of the intervals near zero so that $f(t_i^\star)(G(t_i)-G(t_{i-1}))$ is summable.

One step in the proof is worthy of recording:

Corollary 43.12 *Let $f,g\colon [a,b] \to \mathbb{R}$ be such that $f \in \mathrm{RS}(g,[a,b])$. For each $\epsilon > 0$ there is $\delta > 0$ such that if $\Pi = (\{t_i\}_{i=0}^n,\{t_i^\star\}_{i=1}^n)$ is a marked partition of $[a,b]$ with $\|\Pi\| < \delta$, then*

$$\sum_{i=1}^n \biggl| f(t_i^\star)\bigl[g(t_i)-g(t_{i-1})\bigr] - \int_{t_{i-1}}^{t_i} f\mathrm{d}g \biggr| < \epsilon \tag{43.35}$$

Proof. This follows from (43.33) and a corresponding statement for the negative part. □