42. LEBESGUE AND HENSTOCK-KURZWEIL INTEGRALS

We now briefly sketch what "lies beyond" the Riemann integration theory. Details are of course the subject of follow up courses on analysis.

42.1 Lebesgue integral.

These aforementioned shortcomings of Riemann's theory provided the important motivation for the creation of more advanced theory of *Lebesgue integral*. The main novelty, going perhaps back to the *Cavalieri principle*, is that instead of partitioning the domain of f, we partition the range of f. The easiest way is to do into intervals of length $1/N$; the preimage map is then used to approximate the "area under the graph of f" by

$$\sum_{k\in\mathbb{Z}} \frac{k}{n}\,\text{length}\Bigl(f^{-1}\bigl([\tfrac{k}{n},\tfrac{k+1}{n})\bigr)\Bigr) \tag{42.1}$$

instead of the Riemann sums. (The sum is effectively finite as soon as f is bounded.)

This "minor" philosophical change of approach to integration comes with a significant technical caveat: We need to first develop a robust theory of "length" for sets that are more complicated than just intervals or finite unions thereof. This is another achievements of H. Lebesgue who built his *measure theory* for this purpose.

We have already encountered some aspects of measure theory above and also in the discussion of sets of zero length. Indeed, a natural way to extend the notion of the length to all subsets of $\mathbb{R}$ via

$$\lambda^\star(A) := \inf\Biggl\{\sum_{n=0}^{\infty}\text{length}(I_n)\colon \{I_n\}_{n\in\mathbb{N}} \text{ open intervals } \wedge A\subseteq \bigcup_{n\in\mathbb{N}} I_n\Biggr\} \tag{42.2}$$

(Calling A zero length just meant $\lambda^\star(A)=0$.) Unfortunately, the map $\lambda^\star\colon \mathcal{P}(A)\to[0,\infty]$ lacks one very reasonable property which is factorization under complements. Namely, for $A\subseteq B$ bounded with B a bounded interval, we would like to have

$$\lambda^\star(A)+\lambda^\star(B\smallsetminus A)=\lambda^\star(B) \tag{42.3}$$

but this fails in general because, roughly speaking, the boundary of A may be counted twice on the left-hand side. A solution is to identify a suitable subset of $\mathcal{P}(B)$ for which (42.3) holds. The resulting "special" sets are then called *measurable*.

As it turns out, the collection of measurable sets is closed under countable unions and complements. Moreover, the function $\lambda^\star$ acts additively on countable unions of disjoint measurable sets which makes the restriction of $\lambda^\star$ to the measurable sets a *measure*. Functions that preimage intervals into measurable sets are called *measurable functions*; as it turns out, if a measurable function is also bounded then the sums in (42.1) converge as $n\to\infty$ and define the Lebesgue integral. However, boundedness is not required: measurable functions for which the limit exists are called *Lebesgue integrable*.

The measure theoretic approach to integration turns out to be extremely versatile allowing us to integrate real-valued functions over far more complicated sets than just the reals; e.g., over subsets of $\mathbb{R}^d$, spaces of functions, linear operators, graphs, metric spaces, etc. In spite of this generality, even that theory is not void of shortcomings that we saw for the Riemann integral. Indeed, while handling quite properly pointwise convergence, the Fundamental Theorem of Calculus, part II, still fails to hold without additional provisos — namely, the derivative has to be absolutely integrable. (This integrals in (41.17) and (41.18) thus do not exist in Lebesgue theory either.)

Other approaches to integration have therefore been considered, e.g., by Saks, Perron, Ward, and Lusin culminating in the work of J. Kurzweil and, independently, R. Henstock that we will discuss next.

42.2 Henstock-Kurzweil integral.

Suppose we put ourselves to the task of extending the Riemann integral to a larger class of functions while preserving the framework that extracts the integral as a limit of Riemann sums over marked partitions. In order to allow for unbounded functions, we clearly cannot work with arbitrary marked points but have to allow only those that lie in partition intervals of small-enough size. This leads to the following concept:

Definition 42.1 (Gauge function and associated partitions) *Let $a < b$ be reals. Given function $\gamma\colon [a,b] \to (0,\infty)$, referred to as a* gauge function *in this context, a marked partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ of $[a,b]$ is said to be γ-*fine *if*

$$\forall i = 1,\dots,n\colon [t_{i-1},t_i] \subseteq \big[t_i^\star - \gamma(t_i^\star), t_i^\star + \gamma(t_i^\star)\big]. \tag{42.4}$$

We will now examine the Riemann sums over partitions that are γ-fine for a given gauge function γ. However, for this we need to know that each gauge function γ admits at least one γ-fine marked partition. This will follow from:

Theorem 42.2 (P. Cousin, 1895) *Let $a < b$ be reals and let $\mathcal{I}$ be a collection of non-degenerate closed subintervals of $[a,b]$ with the following property: For each $x \in [a,b]$ there is $\delta > 0$ such that all non-degenerate closed intervals $[c,d]$ satisfying*

$$[c,d] \subseteq [a,b] \;\wedge\; x \in [c,d] \;\wedge\; d - c < \delta \tag{42.5}$$

belong to $\mathcal{I}$. Then there is a partition of $[a,b]$ consisting only of intervals in $\mathcal{I}$, i.e., that there are $a = t_0 < t_1 < \dots < t_n = b$ satisfying

$$\forall i = 1,\dots,n\colon\; [t_{i-1},t_i] \in \mathcal{I} \tag{42.6}$$

The proof of Cousin's theorem is left to a homework exercise. To see that it does the job for us, note that, given a gauge function $\gamma\colon [a,b] \to (0,\infty)$, the set

$$\mathcal{I} := \bigcup_{t\in[a,b]} \big\{[c,d] \cap [a,b]\colon t - \gamma(t) \leqslant c < d \leqslant t + \gamma(t) \;\wedge\; c \leqslant t \leqslant d\big\} \tag{42.7}$$

contains all intervals that can possibly appear in γ-fine partitions of $[a,b]$ and that the set has the property stated in Theorem 42.2.

We remark that Cousin's motivation for the above result was to extend the Heine-Borel theorem from countable covers to arbitrary covers. However, as the wiki page on this result points out: "...Pierre Cousin did not receive any credit. Cousin's theorem was generally attributed to H. Lebesgue as the Borel-Lebesgue theorem. Lebesgue was aware of this result in 1898, and proved it in his 1903 dissertation."

Returning to our main line of thought, we now put forward:

Definition 42.3 (Henstock-Kurzweil integral) *A function* $f\colon [a,b] \to \mathbb{R}$ *is said to be* Henstock-Kurzweil (HKI) integrable *if*

$$\exists L \in \mathbb{R}\, \forall \epsilon > 0\, \exists \gamma \in (0,\infty)^{[a,b]}\, \forall \Pi = \gamma\text{-fine partition}\colon \big|R(f,\Pi) - L\big| < \epsilon. \tag{42.8}$$

The quantity L*, which is unique if exists at all, is the* Henstock-Kurzweil integral.

Remark 42.4 Recall that the marked partition needs that the marked point lie inside the corresponding intervals; i.e., $\forall i = 1,\dots,n\colon t_i^\star \in [t_{i-1}, t_i]$. If this requirement is dropped (while of course keeping that in (42.4)), we gets the so called *McShane integral*. Dropping restrictions extends the set of allowed marked partitions and thus makes fewer functions integrable. As it turns out, McShane's integerl is just equivalent to the Lebesgue integral albeit without ever using the concept of a measure.

Note that for a constant gauge function, the Henstock-Kurzweil integral degenerates to the Riemann integral and so we immediately have:

Lemma 42.5 *Let* $f\colon [a,b] \to \mathbb{R}$ *be a function. Then*

$$f \text{ RI on } [a,b] \;\Rightarrow\; f \text{ HKI on } [a,b] \tag{42.9}$$

Proof. If δ is related to ϵ as in the definition of Riemann integrability, we chose $\gamma(t) := \delta/2$ and observe that every γ-fine partition has mesh at most δ. □

But there are many functions for which the reverse implication fails. The most elementary example is that of the Dirichlet function $1_\mathbb{Q}$ which we have shown is not Riemann integrable. Yet we have:

Lemma 42.6 *For all* $a < b$ *real, the Dirichlet function* $1_\mathbb{Q}$ *is Henstock-Kurzweil interable on* $[a,b]$ *and* $\int_a^b 1_\mathbb{Q}(x)\mathrm{d}x = 0$.

Proof. Let $a < b$. Fix $\epsilon > 0$ and let $\{q_n\}_{n\in\mathbb{N}}$ enumerate all rationals in $[a,b]$. Define

$$\gamma(t) := \begin{cases} \epsilon 2^{-n}, & \text{if } t = q_n \text{ for some } n \in \mathbb{N}, \\ 1, & \text{if } t \in [a,b] \smallsetminus \mathbb{Q}. \end{cases} \tag{42.10}$$

Then for any γ-fine partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ of $[a,b]$,

$$\big|R(f,\Pi)\big| \leqslant \sum_{i=1}^n 1_\mathbb{Q}(t_i^\star)|t_i - t_{i-1}| \leqslant \sum_{i=1}^n 1_\mathbb{Q}(t_i^\star) 2\gamma(t_i^\star) \leqslant \sum_{n=0}^\infty \epsilon 2^{-n+1} = 4\epsilon \tag{42.11}$$

where the first inequality is the triangle inequality for the absolute value, the second inequality follows from (42.4) and the third inequality from the definition of γ. This proves that (42.8) holds with $L := 0$. □

The Henstock-Kurzweil integral (and HK integrability) is checked to be linear in the integrand and additive in the integration domain. We leave these proofs (which are very similar to those for the Riemann integral) to the reader. What is more interesting is that the HK integral allows for unbounded integrands which, roughly speaking, is ensured by choosing the gauge that obey

$$\forall t \in [a,b]\colon\; \gamma(t) \leqslant \frac{\epsilon}{1 + |f(t)|} \tag{42.12}$$

This assumption also seamlessly guarantees that all partition intervals contribute at most ϵ to the Riemann sum regardless on how large f is at the marked point. For similar reasons the HK integral also fairs much better under convergence of its limits thus eliminating the need to consider its "improper" mutations:

Theorem 42.7 (Heke's theorem) *Suppose $a < b$ and $f\colon [a,b] \to \mathbb{R}$ are such that*

$$\forall c \in (a,b]\colon\ f \text{ HKI on } [c,d] \tag{42.13}$$

and

$$\lim_{c\to a^+} \int_c^b f(x)\mathrm{d}x \ \text{ exists} \tag{42.14}$$

Then

$$f \text{ HKI on } [a,b] \ \wedge\ \int_a^b f(x)\mathrm{d}x = \lim_{c\to a^+}\int_c^b f(x)\mathrm{d}x \tag{42.15}$$

In (42.14) *it suffices if the limit is taken along a decreasing sequence.*

As a consequence of this and Lemma 42.5 we get that the improper Riemann integrals (41.16) and (41.17) exist as proper Henstock-Kurzweil integrals with the value as computed by the limits. We leave a detailed proof to a homework exercise.

In order to further demonstrate the strength of the Henstock-Kurzweil integration theory, we now show that the second Fundamental Theorem of Calculus holds in this theory with no provisos on the derivative except for its existence:

Theorem 42.8 (FTCII for Henstock-Kurzweil integral) *Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Choosing $f'(a)$ and $f'(b)$ arbitrarily, f' is then Henstock-Kurzweil integrable on $[a,b]$ and*

$$\int_a^b f'(x)\mathrm{d}x = f(b) - f(a) \tag{42.16}$$

where the integral is the Henstock-Kurzweil sense.

Proof. For each $t \in (a,b)$, the existence of $f'(t)$ shows that

$$\gamma(t) := \frac12 \sup\left\{\delta \in (0,1)\colon \sup_{\substack{x\in[a,b]\\ 0<|x-t|<\delta}} \left|\frac{f(x)-f(t)}{x-t} - f'(t)\right| < \frac{\epsilon}{b-a}\right\} \tag{42.17}$$

is positive and finite. Choosing $f'(a)$ and $f'(b)$ arbitrarily, set also

$$\gamma(a) := \min\left\{\frac{1}{b-a}\,\frac{\epsilon}{1+|f'(a)|}, \frac12 \sup\left\{\delta \in (b-a)\colon \sup_{x\in[a,a+\delta]} \bigl|f(x)-f(a)\bigr| < \frac{\epsilon}{b-a}\right\}\right\} \tag{42.18}$$

and similarly

$$\gamma(b) := \min\left\{\frac{1}{b-a}\,\frac{\epsilon}{1+|f'(a)|}, \frac12 \sup\left\{\delta \in (b-a)\colon \sup_{x\in[b-\delta,b]} \bigl|f(x)-f(a)\bigr| < \frac{\epsilon}{b-a}\right\}\right\} \tag{42.19}$$

For each $t, x \in [a,b]$ we then have

$$x \in \bigl[t-\gamma(t), t+\gamma(t)\bigr] \;\Rightarrow\; \bigl|f(x)-f(t)-f'(t)(t-x)\bigr| < \frac{\epsilon}{b-a}|x-t| \tag{42.20}$$

Let $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ be a γ-fine partition of $[a,b]$. Then the same calculation as in the proof of Theorem 40.7 shows

$$\begin{aligned}
\bigl|f(b)-f(a)-R(f',\Pi)\bigr| &\leqslant \sum_{i=1}^n \bigl|f(t_i)-f(t_{i-1})-f'(t_i^\star)(t_i-t_{i-1})\bigr| \\
&\leqslant \sum_{i=1}^n \bigl|f(t_i)-f(t_i^\star)-f'(t_i^\star)(t_i-t_i^\star)\bigr| + \sum_{i=1}^n \bigl|f(t_{i-1})-f(t_i^\star)-f'(t_i^\star)(t_{i-1}-t_i^\star)\bigr| \qquad (42.21)\\
&\leqslant \sum_{i=1}^n \frac{\epsilon}{b-a}(t_i-t_i^\star) + \sum_{i=1}^n \frac{\epsilon}{b-a}(t_i^\star-t_{i-1}) = \frac{\epsilon}{b-a}\sum_{i=1}^n (t_i-t_{i-1}) = 2\epsilon
\end{aligned}$$

It follows that f' is Henstock-Kurzweil integrable with $\int_a^b f'(x)\mathrm{d}x = f(b)-f(a)$. □

Every Henstock-Kurzweil integrable function is measurable (which we recall means that a preimage of a measurable set is measurable), but the Henstock-Kurzweil integral is actually more general than the Lebesgue integral. Indeed, we have:

Theorem 42.9 *A function $f\colon [a,b] \to \mathbb{R}$ is Lebesgue integrable on $[a,b]$ if and only if f and $|f|$ are Henstock-Kurzweil integrable.*

As noted above, the function $x \mapsto \frac{\sin(1/x)}{x}$ treated in (41.17) using the improper Riemann integral is (properly) Henstock-Kurzweil integrable yet (not being absolutely integrable) it is also not Lebesgue integrable. This is because in Lebesgue's theory of integration, a measurable f is integrable if and only if $|f|$ is integrable. (The issue is unboundedness; for bounded functions on compact intervals, the Lebesgue and HK integrability are the same.) That being said, the Lebesgue theory is superior in its generality of the underlying space over which the integral is taken. This is why it is the dominant integration theory used presently throughout mathematics.