40. Fundamental Theorem of Calculus

Having developed the theory of Riemann integral, we move to the connection between differentiation and integration known under the banner Fundamental Theorem of Calculus. In Newton/Leibnitz' theory, this connection relies on the following concept:

Definition 40.1 *Let $f\colon [a,b] \to \mathbb{R}$. A function $F\colon [a,b] \to \mathbb{R}$ is is said to be an* antiderivative *of f, if F is continuous on $[a,b]$, differentiable on (a,b) and*

$$\forall x \in (a,b)\colon \quad F'(x) = f(x) \tag{40.1}$$

(Another name used for antiderivative is primitive *function.)*

We then say that, whenever a function $f\colon [a,b] \to \mathbb{R}$ admits an antiderivative F on $[a,b]$, the *Newton integral* is defined by

$$\int_a^b f(x)\mathrm{d}x := F(x)\Big|_a^b := F(b) - F(a) \tag{40.2}$$

and, in particular, f is Newton-integrable if it admits an antiderivative. (Note that if $F,G\colon [a,b] \to \mathbb{R}$ are antiderivatives of f on $[a,b]$ then $(F-G)'(x) = f(x) - f(x) = 0$ for all $x \in (a,b)$ and Rolle's Mean-Value Theorem implies that $F - G$ is constant and so the right-hand side of (40.2) is independent of the choice of the antiderivative.)

With this definition of the integral, both statements of the Fundamental Theorem of Calculus follow readily:

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x) \ \wedge \ \int_a^b F'(x)\mathrm{d}x = F(b) - F(a) \tag{40.3}$$

In Newton's integration theory, these are mathematically correct albeit not really deep statements. Indeed, the Newton integral is basically *defined* to have these properties TRUE automatically.

40.1 Differentiating the integral.

In Riemann's integration theory, the integral is defined with no *a priori* connection to the derivative and so the above become theorems that need suitable qualifiers. We start with a question: Suppose f is Riemann integrable on $[a,b]$. Then $F(x) := \int_a^x f(t)\mathrm{d}t$ is well defined for all $x \in [a,b]$. What kind of regularity can we expect from F? We already know that F is continuous but we can say a bit more:

Lemma 40.2 *Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$. Then F defined for all $x \in (a,b]$ by $F(x) := \int_a^x f(t)\mathrm{d}t$ and by $F(a) := 0$ is Lipschitz continuous on $[a,b]$.*

Proof. Let $x,y \in [a,b]$ obey $x < y$. The additivity of the Riemann integral proved in Lemma 36.7 then shows

$$F(y) - F(x) = \int_a^y f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t = \int_x^y f(t)\mathrm{d}t \tag{40.4}$$

By Lemma 36.9, f Riemann integrable implies that f is bounded and so

$$\big|F(y) - F(x)\big| \leqslant \left|\int_x^y f(t)\mathrm{d}t\right| \leqslant \Big(\sup_{t\in[a,b]} \big|f(t)\big|\Big)|y-x| \tag{40.5}$$

Hence F is Lipschitz continuous as claimed. □

Lipschitz continuous functions are not necessarily differentiable (although they do turn out to be differentiable at "most" points). An example of such a Lipschitz continuous function is $F(x) := |x|$ for which we have

$$|x| = \int_0^x \Bigl(1_{[0,\infty)}(t) - 1_{(-\infty,0)}(t)\Bigr)\mathrm{d}t \tag{40.6}$$

The only point of non-differentiability thus coincides with a discontinuity point of the integrand. This is no coincidence in light of:

Lemma 40.3 *Let f be Riemann integrable on $[a,b]$ and set $F(x) := \int_a^x f(t)\mathrm{d}t$. Then*

$$\forall x \in (a,b)\colon \quad f \text{ continuous at } x \;\Rightarrow\; F'(x) \text{ exists} \wedge F'(x) = f(x) \tag{40.7}$$

At $x = a$ the same holds if continuity is replaced by right continuity and derivative by the right derivative, and similarly for left continuity/derivative at $x = b$.

Proof. Let $x \in [a,b)$. Then for all $y \in (x,b)$,

$$F(y) - F(x) - f(x)(y-x) = \int_x^y \bigl[f(t) - f(x)\bigr]\mathrm{d}t \tag{40.8}$$

which implies

$$\Bigl|\frac{F(y)-F(x)}{y-x} - f(x)\Bigr| \leqslant \frac{1}{|y-x|}\Bigl|\int_x^y \bigl[f(t)-f(x)\bigr]\mathrm{d}t\Bigr| \leqslant \sup_{t\in[x,y]}\bigl|f(t)-f(x)\bigr| \tag{40.9}$$

Assuming that f is right-continuous at x, the right-hand side tends to zero as $y \to x^+$ thus proving that the right-derivative of F at x equals $f(x)$. The left-derivative at $x \in (a,b]$ is handled similarly. □

This now shows:

Theorem 40.4 (FTC I) *Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Then $x \mapsto \int_a^x f(t)\mathrm{d}t$ is differentiable on (a,b) — including one-sided derivatives at a and b — and*

$$\forall x \in (a,b)\colon \quad \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x) \tag{40.10}$$

Proof. This follows by applying Lemma 40.3 at all $x \in (a,b)$. □

The restriction to continuous integrand is done merely for convenience of expression. In particular, the integrand need not be continuous at the point where the integral to be differentiable. Indeed, if f only admits a limit at x then the argument (40.8–40.9) still works albeit with $f(x)$ replaced by the limit. But even the existence of the limit is not required for differentiability of the integral. For instance, the function

$$f(x) := \begin{cases} 1, & \text{if } \exists n \in \mathbb{N}\colon\; x = \frac{1}{n+1}, \\ 0, & \text{else}, \end{cases} \tag{40.11}$$

has $F(x) := \int_0^x f(t)\mathrm{d}t = 0$ and so $F'(x)$ exists (and equals zero) at all $x \in \mathbb{R}$ including $x = 0$ where the right limit of f does not exist. Another example (which uses so far undefined but standard functions $\sin(x)$ and $\cos(x)$) is the function f defined by

$$f(x) := \frac{\mathrm{d}}{\mathrm{d}x} x^2 \sin(1/x) = \cos(1/x) - 2x\sin(1/x) \tag{40.12}$$

for $x \neq 0$ and $f(0) = 0$. This function is Riemann integrable on $[0, b]$ for any $b > 0$ and thus $F(x) := \int_0^x f(t)\mathrm{d}t$ is well defined for all $x \in \mathbb{R}$. Since f is continuous away from zero, we have $F'(x) = f(x)$ for all $x \neq 0$ and (by Theorem 40.7 below and a simple limit argument) we in fact have $F(x) = x^2 \sin(1/x)$ at all $x \in \mathbb{R}$. Then $F'(0) = 0$ yet f does not even have a limit as $x \to 0^{\pm}$.

It is worth noting that our previous characterizations of Riemann integrability tell us that integrals of Riemann integrable functions are differentiable at most points:

Corollary 40.5 *Let $f\colon [a, b] \to \mathbb{R}$ be Riemann integrable and let $F\colon [a, b] \to \mathbb{R}$ be defined by $F(t) := \int_a^t f(x)\mathrm{d}x$. Then*

$$\bigl\{x \in (a, b)\colon F'(x) \text{ exists} \,\wedge\, F'(x) = f(x)\bigr\} \tag{40.13}$$

is dense in $[a, b]$ and, in fact, the complement of a set of zero length.

Proof. That the set is the complement of a zero length set follows from Theorem 39.3 and Lemma 40.3. Such a set is automatically dense. A direct argument for density can be based on Lemma 39.1 whose proof is annotated in homework. □

Theorem 40.4 applies also to the lower limits. Indeed, we have:

Lemma 40.6 *Let $f\colon [a, b] \to \mathbb{R}$ be continuous. Then*

$$\forall x \in (a, b)\colon\ \frac{\mathrm{d}}{\mathrm{d}x} \int_x^b f(t)\mathrm{d}t = -f(x) \tag{40.14}$$

Moreover, for all $g, h\colon \mathbb{R} \to [a, b]$ that are differentiable at $x \in \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{g(x)}^{h(x)} f(t)\mathrm{d}t = f\bigl(h(x)\bigr)h'(x) - f\bigl(g(x)\bigr)g'(x) \tag{40.15}$$

We leave the proof of this lemma to a homework exercise.

40.2 Integrating the derivative.

We now move to the second part of the Fundamental Theorem of Calculus, which concerns integrals of derivatives. The precise statement is as follows:

Theorem 40.7 (FTC II) *Let $a < b$ be reals and let $F\colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then*

$$F' \text{ Riemann integrable on } [a, b] \ \Rightarrow\ \int_a^b F'(x)\mathrm{d}x = F(b) - F(a) \tag{40.16}$$

Proof. Pick $\epsilon > 0$. Assuming F' to be Riemann integrable, there is $\delta > 0$ such that for all partitions Π of $[a, b]$ with $\|\Pi\| < \delta$ we have

$$\left| R(F', \Pi) - \int_a^b F'(x)\mathrm{d}x \right| < \epsilon \tag{40.17}$$

Fixing this $\delta > 0$, let $n \in \mathbb{N}$ be such that $n\delta > b - a$ and define $t_i := a + \frac{i}{n}(b-a)$ for all $i = 0, \dots, n$. By Lagrange's Mean-Value Theorem, for each $i = 1, \dots, n$,

$$\exists t_i^\star \in [t_{i-1}, t_i]\colon \quad F(t_i) - F(t_{i-1}) = F'(t_i^\star)(t_i - t_{i-1}) \tag{40.18}$$

Picking one such $t_i^\star$ in each $[t_{i-1}, t_i]$, the marked partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ then has mesh less than δ and obeys

$$F(b) - F(a) = \sum_{i=1}^n \bigl(F(t_i) - F(t_{i-1})\bigr) = \sum_{i=1}^n F'(t_i^\star)(t_i - t_{i-1}) = R(F', \Pi) \tag{40.19}$$

Using (40.17) it follows that

$$\left| F(b) - F(a) - \int_a^b F'(x)\,\mathrm{d}x \right| < \epsilon \tag{40.20}$$

As this holds for all $\epsilon > 0$, we have the conclusion of (40.16). □

The statements Theorems 40.4 and 40.7 give us the precise versions of (40.3) in Riemann's theory. In summary, they say that

- derivative inverts Riemann integrals of all continuous functions, and
- Riemann integral inverts all Riemann integrable derivatives.

Neither inversion is thus perfect, unlike for the Newton integral which worked whenever the function could be antidifferentiated. We will return to these shortcomings in Sections 41 and **??**.

40.3 Applications.

The Fundamental Theorem of Calculus, albeit somewhat restricted in Riemann's theory, is the basic tool for computing integrals. Other tools exist that generally convert one integral to another. Here is one frequently used:

Corollary 40.8 (Integration by parts) *Suppose $f, g\colon [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b) such that f' and g' — with values at a and b chosen arbitrarily — are Riemann integrable on $[a,b]$. Then*

$$\int_a^b f'(x)g(x)\,\mathrm{d}x = f(b)g(b) - f(a)g(a) - \int_a^b g'(x)f(x)\,\mathrm{d}x \tag{40.21}$$

Proof. The Product Rule for derivative shows that $f \cdot g$ is differentiable and, under our condition, $(f \cdot g)' = f'g + g'f$ is Riemann integrable. Theorem 40.7 shows

$$\int_a^b \bigl[f'(x)g(x) + g'(x)f(x)\bigr]\mathrm{d}x = \int_a^b (f \cdot g)'(x)\,\mathrm{d}x = f(b)g(b) - f(a)g(a) \tag{40.22}$$

Since $f'g$ and $g'f$ are individually Riemann integrable, the integral on the left-hand side can be written as the sum of two Riemann integrals. □

Another standard method for converting one integral to another is:

Corollary 40.9 (Substitution Rule) *Let $c < d$ and $a < b$ be reals and assume $f\colon [c,d] \to \mathbb{R}$ and $\varphi\colon [a,b] \to (c,d)$ are functions such that:*

(1) *φ is continuous on $[a,b]$ and differentiable on (a,b),*

(2) *f is continuous on $[c,d]$,*
(3) *$(f\circ\varphi)\cdot\varphi'$ is Riemann integrable on $[a,b]$.*

Then

$$\int_{\varphi(a)}^{\varphi(b)} f(t)\mathrm{d}t = \int_a^b f\bigl(\varphi(x)\bigr)\varphi'(x)\mathrm{d}x \tag{40.23}$$

Proof. Since f is continuous, $F(x) := \int_c^x f(t)\mathrm{d}t$ is well defined and, by Theorem 40.4, obeys $F'(t) = f(t)$ for all $t\in(c,d)$. Hence also the derivative of $F\circ\varphi$ equals the product $(f\circ\varphi)\cdot\varphi'$. Theorem 40.7 then equates both sides of (40.23) with $F(\varphi(b)) - F(\varphi(a))$. □

A more substantive application of FTC is the content of:

Theorem 40.10 (Taylor theorem with remainder) *Let $a<b$ be reals and $f\colon(a,b)\to\mathbb{R}$ an $(n+1)$-times differentiable function, for some $n\in\mathbb{N}$. Assume $f^{(n+1)}$ is Riemann integrable on any closed subinterval of (a,b). Then*

$$\forall x,x_0\in(a,b)\colon\quad f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{n!}\int_{x_0}^x f^{(n+1)}(z)(x-z)^n\mathrm{d}z \tag{40.24}$$

Proof. We will prove this by induction on n. For $n=0$ the statement (40.24) is just Theorem 40.7 (which requires only that f' is Riemann integrable). Assume therefore that the statement holds for some n and let $f\colon(a,b)\to\mathbb{R}$ be now $(n+2)$-times differentiable with $f^{(n+2)}$ Riemann integrable. Abbreviating the polynomial on the right of (40.24) as $P_n(x)$, from the statement for n we then have

$$f(x) = P_n(x) + \frac{1}{n!}\int_{x_0}^x f^{(n+1)}(z)(x-z)^n\mathrm{d}z \tag{40.25}$$

Integration by parts; namely, Corollary 40.8 with $g(z) := \frac{1}{n+1}(x-z)^{n+1}$ then shows

$$\begin{aligned}
&\frac{1}{n!}\int_{x_0}^x f^{(n+1)}(z)(x-z)^n\mathrm{d}z\\
&\quad= \frac{1}{n!}\int_{x_0}^x f^{(n+1)}(z)g'(z)\mathrm{d}z\\
&\quad= \frac{1}{n!}f^{(n+1)}(z)g(z)\Big|_{x_0}^x - \frac{1}{n!}\int_{x_0}^x f^{(n+2)}(z)g(z)\mathrm{d}z\\
&\quad= \frac{1}{(n+1)!}f^{(n+1)}(x_0)(x-x_0)^{n+1} + \frac{1}{(n+1)!}\int_{x_0}^x f^{(n+2)}(z)(x-z)^{n+1}\mathrm{d}z
\end{aligned} \tag{40.26}$$

Noting that the first term on the right equals $P_{n+1}(x) - P_n(x)$, we have proved (40.24) for n replaced by $n+1$. By induction, the claim holds for all $n\in\mathbb{N}$. □

The statement (40.24) should be compared with the pointwise version of Taylor's theorem in which the error $f(x) - P_n(x)$ takes the form $\frac{1}{(n+1)!}f^{(n+1)}(\xi)(x-x_0)^{n+1}$ for some ξ between x_0 and x. This term usually fares similarly when uniform estimates on the error are required but has the disadvantage of being dependent on an unknown intermediate point ξ. The error in (40.24) is expressed as an explicit integral and is thus better suited when further manipulations with the error term are required.