38. SUFFICIENT CONDITIONS FOR RIEMANN INTEGRABILITY

As noted above, one principal novelty of Riemann's approach to integration is to make the existence of the integral a regularity property of the function in its own right. However, this still leaves us with the need for easily checkable sufficient conditions for Riemann integrability. In this section we go over a list of progressively more demanding sufficient conditions. This will reveal ideas that will play an important role in our full characterization of Riemann integrability in the next section.

38.1 Continuous functions and variations thereof.

One of the earlier attempts to make the concept of integral well defined was put forward by Cauchy, who insisted on working with continuous integrands. It is reassuring that Cauchy's treatment becomes subsumed by Riemann's:

Lemma 38.1 *Let $f\colon [a,b]\to\mathbb{R}$. Then*

$$f \text{ continuous on } [a,b] \;\Rightarrow\; f \text{ Riemann integrable on } [a,b] \tag{38.1}$$

Proof. By the Bolzano-Weierstrass Theorem, a continuous $f\colon [a,b]\to\mathbb{R}$ is automatically bounded and uniformly continuous. This means that

$$\forall\epsilon>0\,\exists\delta>0\,\forall s,t\in[a,b]\colon\; 0<t-s<\delta \;\Rightarrow\; \operatorname{osc}\bigl(f,[s,t]\bigr)<\frac{\epsilon}{b-a} \tag{38.2}$$

Picking such a $\delta>0$, ti follows that for any partition $\Pi=\{t_i\}_{i=1}^n$ of $[a,b]$,

$$\|\Pi\|<\delta \;\Rightarrow\; \sum_{i=1}^n \operatorname{osc}\bigl(f,[t_{i-1},t_i]\bigr)(t_i-t_{i-1})<\frac{\epsilon}{b-a}\sum_{i=1}^n(t_i-t_{i-1})=\epsilon \tag{38.3}$$

As one such a partition can definitely be constructed, Theorem 37.9 implies that f is Riemann integrable. □

The previous lemma notwithstanding, Riemann integral does not at all require the integrand to be continuous. For instance, we have:

Lemma 38.2 *Let $f\colon [a,b]\to\mathbb{R}$ be bounded with a finite number of discontinuities. Then f is Riemann integrable on $[a,b]$ and the values of f at the discontinuity points are immaterial for the Riemann integral.*

Proof. Fix $\epsilon>0$ and let $x_1,\dots,x_n$ enumerate the discontinuities of f. Pick δ' with

$$0<\delta'<\min\Bigl\{\frac12\min_{0\leqslant i<j\leqslant n}|x_i-x_j|,\frac1n\frac{\epsilon}{1+\sup|f|}\Bigr\} \tag{38.4}$$

where $\sup|f|:=\sup_{x\in[a,b]}|f(x)|$. Then f is continuous on

$$A:=[a,b]\smallsetminus\bigcup_{i=0}^n(x_i-\delta',x_i+\delta') \tag{38.5}$$

Since A is closed and, being a subset of a compact set, compact, the Bolzano-Weierstrass Theorem still applies to give us a $\delta''>0$ such that

$$\forall s,t\in A\colon\; 0<t-s<\delta''\;\wedge\;[s,t]\subseteq A \;\Rightarrow\; \operatorname{osc}\bigl(f,[s,t]\bigr)<\frac{\epsilon}{b-a} \tag{38.6}$$

Now pick $m \in \mathbb{N}$ such that $m\delta'' > b - a$ and consider the partition $\Pi = \{t_i\}_{i=1}^N$ consisting of the points $x_i - \epsilon$ and $x_i + \epsilon$ for $i = 1, \dots, n$ and the points of the form $a + j/N$ indexed by $j = 0, \dots, m$ that lie in A. For each $i = 1, \dots, N$, the interval $[t_{i-1}, t_i]$ then coincides with one of $[x_k - \delta', x_k + \delta']$ (because, by our choice of δ', these intervals are disjoint from each other) or is an interval contained in A of length less than δ''. Denote

$$I := \bigl\{i = 1, \dots, N \colon [t_{i-1}, t_i] \subseteq A\bigr\} \tag{38.7}$$

Then (38.6) gives

$$\sum_{i\in I} \operatorname{osc}\bigl(f, [t_{i-1}, t_i]\bigr)(t_i - t_{i-1}) < \frac{\epsilon}{b-a}\sum_{i=1}^{n}(t_i - t_{i-1}) < \frac{\epsilon}{b-a}\sum_{i=1}^{N}(t_i - t_{i-1}) < \epsilon \tag{38.8}$$

while $\delta' < n^{-1}\epsilon/(1 + \sup|f|)$ and $\operatorname{osc}(f, E) \leqslant 2\sup|f|$ and $N \smallsetminus |I| \leqslant n$ give

$$\sum_{i\notin I} \operatorname{osc}\bigl(f, [t_{i-1}, t_i]\bigr)(t_i - t_{i-1}) \leqslant n(2\sup|f|)(2\delta') < \frac{4n\epsilon \sup|f|}{n(1 + \sup|f|)} \leqslant 4\epsilon \tag{38.9}$$

Putting (38.8–38.9) together and invoking (37.21) yields $U(f, \Pi) - L(f, \Pi) < 5\epsilon$. Theorem 37.9 implies that f is Riemann integrable. □

38.2 More intricate examples.

Thinking about the previous proof, a finite number of discontinuities is not at all the limit of what the Riemann integral is able to take. For instance, writing all of the rationals into a sequence $\{q_n\}_{n\in\mathbb{N}}$, the function f defined in (27.23) is Riemann integrable even though it is discontinuous at each rational. This follows from the following more general fact:

Lemma 38.3 *Suppose that* $f \colon [a, b] \to \mathbb{R}$ *is bounded and such that* $\lim_{z\to x} f(z)$ *exists for all* $x \in [a, b]$. *Then* f *is Riemann integrable on* $[a, b]$.

While the proof of this is instructive, we can directly shoot for a stronger result than this and instead state and prove:

Lemma 38.4 *Let* $f \colon [a, b] \to \mathbb{R}$ *be bounded and with no discontinuities of the second kind on* (a, b). *Then* f *is Riemann integrable on* $[a, b]$.

However, once the cardinality of the set of discontinuities is uncountable, the situation is more complicated. An extreme example is:

Lemma 38.5 *The Dirichlet function* $1_{\mathbb{Q}}$ *is not integrable on any bounded closed interval.*

Proof. Since $\operatorname{osc}(1_{\mathbb{Q}}, [s, t]) = 1$ for all real $s < t$, for any partition $\{t_i\}_{i=0}^n$ of $[a, b]$,

$$\sum_{i=1}^{n} \operatorname{osc}\bigl(1_{\mathbb{Q}}, [t_{i-1}, t_i]\bigr)(t_i - t_{i-1}) = b - a \tag{38.10}$$

thus showing that (37.22) is FALSE. □

However, as it turns out, the cardinality itself is not the primary culprit here as our next example shows:

Lemma 38.6 *Recall that the Cantor ternary set is defined as*

$$C := \left\{ \sum_{i=0}^{\infty} \frac{2\sigma_i}{3^{i+1}} : \{\sigma_i\}_{i\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}} \right\} \tag{38.11}$$

Then the function 1_C *defined as*

$$1_C(x) := \begin{cases} 1, & \text{if } x \in C, \\ 0, & \text{if } x \notin C, \end{cases} \tag{38.12}$$

is Riemann integrable on $[0,1]$ *while being discontinuous at all points of* C*, which is uncountable (in fact,* C *is of the cardinality of the continuum).*

We again leave the proof of this lemma to a homework exercise.

To summarize the above observations, what mattered in all the examples above was that the function of concern was bounded and that the set of its discontinuities of size larger than any given positive number could be covered by a finite union of intervals whose total length is less than any given positive number. As we will show in the next section, this is nearly what in fact characterizes the whole class of Riemann integrable functions in full generality.