

37. DARBOUX INTEGRAL

We proceed to give another version of integration theory pioneered by G. Darboux. While this version turns out to be equivalent to Riemann's, some parts of Darboux's approach become useful in their further generalization to the Lebesgue integral (which we will not discuss in this course as it requires the concept of a measure).

37.1 Darboux integrability and integral.

While Riemann's approach to integration relies on the (metric) completeness of the reals, Darboux's approach is based on the ordering property of the reals and the least upper bound axiom. We start with:

Definition 37.1 (Upper/lower Darboux sum) *Let $a < b$ be reals and $f: [a, b] \rightarrow \mathbb{R}$ a bounded function. Given a partition Π consisting of points $\{t_i\}_{i=1}^n$ satisfying (36.1), we define the upper Darboux sum by*

$$U(f, \Pi) := \sum_{i=1}^n \left[\sup_{x \in [t_{i-1}, t_i]} f(x) \right] (t_i - t_{i-1}) \quad (37.1)$$

and its lower Darboux sum by

$$L(f, \Pi) := \sum_{i=1}^n \left[\inf_{x \in [t_{i-1}, t_i]} f(x) \right] (t_i - t_{i-1}) \quad (37.2)$$

We will write these even for Π denoting a marked partition while noting that the sums do not depend on the marked points.

The following is easy to check:

Lemma 37.2 *Let $f: [a, b] \rightarrow \mathbb{R}$ be bounded. Then for any marked partition Π , the associated Riemann sum $R(f, \Pi)$ from (36.4) obeys*

$$L(f, \Pi) \leq R(f, \Pi) \leq U(f, \Pi) \quad (37.3)$$

Moreover, for each $\epsilon > 0$ there are marked partitions Π and Π' such that

$$R(f, \Pi) \geq U(f, \Pi) - \epsilon \quad \wedge \quad R(f, \Pi') \leq L(f, \Pi') + \epsilon \quad (37.4)$$

and such that Π' and Π'' have the same partition points as Π .

A natural way to think of $U(f, \Pi)$ is as the minimal area under the graphs of functions that exceed (or, more precisely, is no less than) f and are constant on each interval $(t_{i-1}, t_i]$. Similarly, $L(f, \Pi)$ is then the maximal area for all such functions that are less (or, more precisely, no larger) than f .

The advantage of the Darboux sums over Riemann's is that they have several useful monotonicity properties. The highlight of these is the content of:

Lemma 37.3 *Let $f: [a, b] \rightarrow \mathbb{R}$ be bounded and given two partitions Π and Π' of $[a, b]$, write $\Pi \subseteq \Pi'$ if all partition points of Π occur among those of Π' . Then for all partitions Π and Π' ,*

$$\Pi \subseteq \Pi' \Rightarrow L(f, \Pi) \leq L(f, \Pi') \leq U(f, \Pi') \leq U(f, \Pi) \quad (37.5)$$

In particular, we have

$$\forall \Pi, \Pi' \text{ partitions of } [a, b]: \quad L(f, \Pi) \leq U(f, \Pi') \quad (37.6)$$

Proof. We start with (37.5). Here it suffices to assume that Π' consists of Π and one point u that lies in interval $[t_{i-1}, t_i]$. In this case

$$\begin{aligned} U(f, \Pi') - U(f, \Pi) &= \left[\sup_{x \in [t_{i-1}, u]} f(x) \right] (u - t_{i-1}) \\ &\quad + \left[\sup_{x \in [u, t_i]} f(x) \right] (t_i - u) - \left[\sup_{x \in [t_{i-1}, t_i]} f(x) \right] (t_i - t_{i-1}) \end{aligned} \quad (37.7)$$

Using that

$$\max \left\{ \sup_{x \in [t_{i-1}, u]} f(x), \sup_{x \in [u, t_i]} f(x) \right\} \leq \sup_{x \in [t_{i-1}, t_i]} f(x) \quad (37.8)$$

we then get $U(f, \Pi') - U(f, \Pi) \leq 0$ as desired. The inequality $L(f, \Pi) \leq L(f, \Pi')$ is proved similarly.

In order to get (37.6) from this, let Π and Π' be arbitrary partitions and let $\Pi \cup \Pi'$ denote the partition obtained by combining the points in Π and Π' . Then $\Pi \cup \Pi' \subseteq \Pi$ and $\Pi \cup \Pi' \subseteq \Pi'$ and so

$$L(f, \Pi) \leq L(f, \Pi \cup \Pi') \leq U(f, \Pi \cup \Pi') \leq U(f, \Pi') \quad (37.9)$$

holds by (37.5). □

We now put forward:

Definition 37.4 (Upper/lower Darboux integral) *Let $f: [a, b] \rightarrow \mathbb{R}$ be bounded. Then*

$$\int_a^{\overline{b}} f(x) dx := \inf \{ U(f, \Pi) : \Pi \text{ is a partition of } [a, b] \} \quad (37.10)$$

is the upper Darboux integral while

$$\int_a^{\underline{b}} f(x) dx := \sup \{ L(f, \Pi) : \Pi \text{ is a partition of } [a, b] \} \quad (37.11)$$

is the lower Darboux integral.

Since $U(f, \Pi)$ is defined using suprema, the upper Darboux integral can be thought of as a version of “limsup” of Riemann sums and, similarly, the lower Darboux integral as a version of “liminf.” As a consequence of (37.6), we get:

Corollary 37.5 *For all bounded $f: [a, b] \rightarrow \mathbb{R}$ we then have*

$$\int_a^{\underline{b}} f(x) dx \leq \int_a^{\overline{b}} f(x) dx \quad (37.12)$$

We can think of this as the analogy of statement that “liminf” is no larger than the “limsup.” The existence of the “limit,” which is what we have (albeit in a different form) associated with Riemann integrability, is then characterized by the “limsup” being equal to the “liminf:”

Definition 37.6 (Darboux integrability/integral) We say that a bounded $f: [a, b] \rightarrow \mathbb{R}$ is Darboux integrable (DI) if

$$\int_a^b f(x)dx = \int_a^{\overline{b}} f(x)dx \tag{37.13}$$

The common value of these quantities is then called the Darboux integral.

37.2 Equivalence of Darboux and Riemann integral.

As announced earlier, the two approaches to integral turn out to be equivalent. We state this along with a convenient necessary and sufficient condition:

Lemma 37.7 (Characterization of Darboux integrability) For all $f: [a, b] \rightarrow \mathbb{R}$ bounded,

$$f \text{ DI on } [a, b] \Leftrightarrow \forall \epsilon > 0 \exists \Pi = \text{partition of } [a, b]: U(f, \Pi) - L(f, \Pi) < \epsilon \tag{37.14}$$

Proof. For any partition Π we have

$$\int_a^{\overline{b}} f(x)dx - \int_a^b f(x)dx \leq U(f, \Pi) - L(f, \Pi) \tag{37.15}$$

With the help of (37.12), this proves \Leftarrow in (37.14).

On the other hand, by the properties of suprema and infima, for each $\epsilon > 0$ there are partitions Π and Π' such that

$$U(f, \Pi) \leq \int_a^{\overline{b}} f(x)dx + \epsilon \quad \wedge \quad L(f, \Pi') \geq \int_a^b f(x)dx - \epsilon. \tag{37.16}$$

In light of (37.9), the common refinement $\Pi'' := \Pi \cup \Pi'$ of Π and Π' obeys

$$\begin{aligned} U(f, \Pi'') - L(f, \Pi'') &\leq U(f, \Pi) - L(f, \Pi') \\ &\leq \int_a^{\overline{b}} f(x)dx - \int_a^b f(x)dx + 2\epsilon = 2\epsilon, \end{aligned} \tag{37.17}$$

where the last equality follow from (37.13) and the assumption that f is Darboux integrable. This proves \Rightarrow in (37.14). □

Note that the previous proof actually shows:

Corollary 37.8 For all $f: [a, b] \rightarrow \mathbb{R}$ bounded,

$$\int_a^{\overline{b}} f(x)dx - \int_a^b f(x)dx = \inf \left\{ U(f, \Pi) - L(f, \Pi) : \Pi = \text{partition of } [a, b] \right\} \tag{37.18}$$

An important point in applications is that the quantity $U(f, \Pi) - L(f, \Pi)$ can be expressed using the concept of (unrestricted) *oscillation*, which for a bounded real-valued function f on a non-empty set $A \subseteq \text{Dom}(f)$ is defined as

$$\text{osc}(f, A) := \sup \{ |f(y) - f(x)| : x, y \in A \} \tag{37.19}$$

(this corresponds to $\text{osc}_f(f, \infty)$ in our earlier notation). To see why this is relevant for us, note that

$$\text{osc}(f, A) = \sup_{x \in A} f(x) - \inf_{x \in A} f(x) \quad (37.20)$$

and so for any $f: [a, b] \rightarrow \mathbb{R}$ and any partition $\Pi = \{t_i\}_{i=0}^n$ of $[a, b]$,

$$U(f, \Pi) - L(f, \Pi) = \sum_{i=1}^n \text{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}). \quad (37.21)$$

The integrability thus (vaguely) corresponds to there being a sequence of partitions along which the total length of intervals where the oscillation is appreciable can be made arbitrarily small. Note that, unlike for the Riemann integral, we make no restrictions as to the mesh of the partition used but, in all but the trivial cases, the partitions Π with small value of $U(f, \Pi) - L(f, \Pi)$ will have small mesh.

The main result of this section is:

Theorem 37.9 (Equivalence of Darboux and Riemann) *For all $f: [a, b] \rightarrow \mathbb{R}$ be bounded:*

(1) *Riemann integrability of f on $[a, b]$ implies*

$$\forall \epsilon > 0 \exists \Pi = \{t_i\}_{i=0}^n \text{ partition of } [a, b]: \sum_{i=1}^n \text{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) < \epsilon \quad (37.22)$$

(2) *Condition (37.22) implies Riemann integrability of f on $[a, b]$.*

In particular, for all bounded or all $f: [a, b] \rightarrow \mathbb{R}$,

$$f \text{ RI on } [a, b] \Leftrightarrow f \text{ DI on } [a, b] \quad (37.23)$$

and, if these are TRUE, then the Darboux and the Riemann integrals coincide.

Proof. Let us start with (1). Fix $\epsilon > 0$. By the arguments underlying the proof of Lemma 37.2, for any choices of partition points, there are choices of marked points such that the associated partitions Π and Π' obey

$$R(f, \Pi) \geq U(f, \Pi) - \epsilon \wedge R(f, \Pi') \leq L(f, \Pi') + \epsilon \quad (37.24)$$

Since f is assumed Riemann integrable, the Cauchy criterion for Riemann integrability (Lemma 36.8) implies that there is $\delta > 0$ such that for any marked partitions $\|\Pi\| < \delta$ and $\|\Pi'\| < \delta$ then

$$|R(f, \Pi) - R(f, \Pi')| < \epsilon \quad (37.25)$$

Picking the partition points so that $\|\Pi\| < \delta$ and $\|\Pi'\| < \delta$ holds and the marked points so that (37.24) is TRUE, we then get

$$\begin{aligned} U(f, \Pi \cup \Pi') - L(f, \Pi \cup \Pi') &\leq U(f, \Pi) - L(f, \Pi') \\ &\leq |R(f, \Pi) - R(f, \Pi')| + 2\epsilon < 3\epsilon \end{aligned} \quad (37.26)$$

where $\Pi \cup \Pi'$ is the common refinement of Π and Π' and where the first inequality follows from (37.9). The claim follows from (37.21) by relabeling 3ϵ as ϵ .

For (2) we pick $\epsilon > 0$ and let $\Pi_0 = \{t_i\}_{i=0}^n$ be a partition of $[a, b]$ such that

$$U(f, \Pi_0) - L(f, \Pi_0) < \epsilon. \quad (37.27)$$

Pick $\delta > 0$ satisfying

$$2n\delta \left(\sup_{x \in [a,b]} |f(x)| \right) < \epsilon \quad (37.28)$$

Given marked partitions Π and Π' such that $\|\Pi\| < \delta$ and $\|\Pi'\| < \delta$, let $\Pi_0 \cup \Pi'$, resp., $\Pi_0 \cup \Pi''$ be marked partitions with the partition points of both Π_0 and Π , resp., both Π_0 and Π' and the marked points of Π' , resp., Π'' in the intervals of $\Pi_0 \cup \Pi'$, resp., $\Pi_0 \cup \Pi''$ where the marked points of these partition fall into and left endpoints of the intervals in those where they do not. Then argument underlying (36.36) then shows

$$|R(f, \Pi_0 \cup \Pi) - R(f, \Pi)| \leq 2n\delta \left(\sup_{x \in [a,b]} |f(x)| \right) < \epsilon \quad (37.29)$$

and, similarly, also

$$|R(f, \Pi_0 \cup \Pi') - R(f, \Pi')| \leq 2n\delta \left(\sup_{x \in [a,b]} |f(x)| \right) < \epsilon \quad (37.30)$$

But (37.3) and (37.5) give

$$L(f, \Pi_0) \leq L(f, \Pi_0 \cup \Pi) \leq R(f, \Pi_0 \cup \Pi) \leq U(f, \Pi_0 \cup \Pi) \leq U(f, \Pi_0) \quad (37.31)$$

and similarly for $R(f, \Pi_0 \cup \Pi')$. Therefore,

$$\begin{aligned} |R(f, \Pi) - R(f, \Pi')| &\leq 2\epsilon + \left| R(f, \Pi_0 \cup \Pi) - R(f, \Pi_0 \cup \Pi') \right| \\ &\leq 2\epsilon + U(f, \Pi_0) - L(f, \Pi_0) < 3\epsilon \end{aligned} \quad (37.32)$$

We have thus verified the Cauchy criterion from Lemma 36.8 and proved that f is Riemann integrable.

With f both Riemann and Darboux integrable, we use Lemma 37.7 to find a sequence of marked partitions $\{\Pi_n\}_{n \in \mathbb{N}}$ such that $U(f, \Pi_n) - L(f, \Pi_n) \rightarrow 0$. Thanks to monotonicity under insertion of points, we may assume $\|\Pi_n\| \rightarrow 0$. Since the sequences $\{U(f, \Pi_n)\}_{n \in \mathbb{N}}$ and $\{L(f, \Pi_n)\}_{n \in \mathbb{N}}$ are bounded, choosing a subsequence we may assume that $I := \lim_{n \rightarrow \infty} U(f, \Pi_n)$ and $I' := \lim_{n \rightarrow \infty} L(f, \Pi_n)$ exist. But then $I' = I$ and I equals to the Darboux integral of f on $[a, b]$. The inequality (37.3) in turn forces $R(f, \Pi_n) \rightarrow I$. Since $\|\Pi_n\| \rightarrow 0$, this implies that I is also the Riemann integral of f on $[a, b]$. \square

37.3 Consequences.

Since the two concepts of integrability are thus shown to be equivalent, we will henceforth refer to both Darboux's and Riemann's integral simply as *the* Riemann integral. Darboux's integral is often simpler to work with conceptually as it easier to verify its main criterion for integrability (37.22). (Recall that the harder part of the proof of Theorem 37.9 was the implication "(37.22) \Rightarrow Riemann integrability.") This gives us a far simpler proof of the additivity with respect to the domain of integration:

Proof of Lemma 36.7 again. Let $a < c < b$. If f is Darboux integrable on $[a, c]$ and $[c, b]$, given $\epsilon > 0$ there exists partitions Π' and Π'' of these intervals, respectively, such that

$$U(f, \Pi') - L(f, \Pi') < \epsilon \quad \wedge \quad U(f, \Pi'') - L(f, \Pi'') < \epsilon \quad (37.33)$$

Let $\Pi := \Pi \circ \Pi'$ be the concatenation of the two partitions. Then

$$\begin{aligned} U(f, \Pi) &= U(f, \Pi') + U(f, \Pi') \\ L(f, \Pi) &= L(f, \Pi') + L(f, \Pi') \end{aligned} \quad (37.34)$$

and so adding the above inequalities we get

$$U(f, \Pi) - L(f, \Pi) < 2\epsilon \quad (37.35)$$

proving Darboux integrability of f on $[a, b]$.

Conversely, if f is Darboux integrable on $[a, b]$, then there is a partition of Π for which $U(f, \Pi) - L(f, \Pi) < \epsilon$. The monotonicity under insertion of a point allows us to assume that Π contains c as a partition point. But then $\Pi = \Pi' \circ \Pi''$ for partitions Π' and Π'' of $[a, c]$ and $[c, b]$, respectively, which by (37.34) must satisfy (37.33), proving Darboux integrability of f on $[a, c]$ and $[c, b]$. \square

The Darboux integrability criterion is very useful in deriving further new facts:

Lemma 37.10 *Let $f: [a, b] \rightarrow \mathbb{R}$ be bounded. Then*

(1) *f Riemann integrable implies*

$$|f| \text{ Riemann integrable} \wedge \left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx \quad (37.36)$$

(2) *f, g Riemann integrable $\Rightarrow f \cdot g$ Riemann integrable*

Proof. This follows from the easy bounds

$$\text{osc}(|f|, A) \leq \text{osc}(f, A) \quad (37.37)$$

and

$$\text{osc}(f \cdot g, A) \leq \left(\sup_{x \in [a, b]} |f(x)| \right) \text{osc}(g, A) + \left(\sup_{x \in [a, b]} |g(x)| \right) \text{osc}(f, A) \quad (37.38)$$

and, in the second case, also the fact that $\Pi \mapsto U(f, \Pi) - L(f, \Pi)$ does not increase under refinement of the partition. The inequality in (1) is implied by $|R(f, \Pi)| \leq R(|f|, \Pi)$. We leave the remaining details to the reader. \square

The proof of (1) can in fact be generalized further to get:

Lemma 37.11 *Let $f: [a, b] \rightarrow \mathbb{R}$ be Riemann integrable on $[a, b]$ and let $F: \mathbb{R} \rightarrow \mathbb{R}$ be defined and continuous on $\overline{\text{Ran}(f)}$. Then $F \circ f$ is Riemann integrable on $[a, b]$.*

Leaving the proof of this lemma to homework we note that, on the other hand, some statements are slightly harder to show in Darboux's approach than in Riemann's approach. One of these is additivity with respect to the integrand which in the context of Darboux integral takes the following form:

Lemma 37.12 *Let $a < b$ be reals and $f, g: [a, b] \rightarrow \mathbb{R}$ bounded functions. Then the upper Darboux integral is subadditive,*

$$\int_a^b (f + g)(x) dx \leq \int_a^b f(x) dx + \int_a^b g(x) dx \quad (37.39)$$

and the lower Darboux integral is superadditive,

$$\int_a^b (f + g)(x) dx \geq \int_a^b f(x) dx + \int_a^b g(x) dx \quad (37.40)$$

In particular, if both f and g are Darboux integrable, then so is $f + g$ and the Darboux integral of $f + g$ is the sum of Darboux integrals of f and g .

We leave the proof of this lemma to homework while noting that one can also construct non-Darboux integrable f and/or g such that the inequalities in (37.39–37.40) are strict. The upper/lower Darboux integrals are also positive homogeneous, meaning that the upper integral of λf is λ -multiple of the upper integral of f for all $\lambda \geq 0$. On the other hand, multiplying f by negative λ transforms the upper integral of λf to the λ -multiple of the lower integral and *vice versa*.

The main disadvantage of Darboux's approach is that is based on the ordering property of \mathbb{R} , which is unavailable as soon as the integrand takes value in more complicated sets (such as the generalization of the Riemann integral to vector-valued functions) or we need to work with the generalization of the Riemann integral to what we will later call the Stieltjes integral.