36. RIEMANN INTEGRAL

An important early achievement of the rigorous approach to mathematics in mid 19th century was the definition of (what we now call) the Riemann integral. We will follow the original ideas of Riemann while the textbook focuses on the approach due to Darboux which, as we will show later, is equivalent to Riemann's. Both approaches have their own merit on their own as well as in various extensions.

36.1 Area under a curve.

The problem of finding "area bounded by a curve" dates back to ancient Greece. Even there it was already understood that one should proceed by way of exhaustion via polygonal approximations. This is what we will do using the concept of the integral which, while introduced already by Newton and Leibnitz in their treatments of differential and integral calculus and studied by Cauchy later, was treated properly only by Riemann in his 1854 Habilitationschrift.

A common idea how to compute, or more precisely *define*, the "area under the graph of a function" was to approximate it by the area of many thin (or even infinitesimal) rectangles. Riemann formalized this precisely using the following concepts:

Definition 36.1 (Marked partition and Riemann sum) *Let $a < b$ be reals. A* marked partition Π *of interval $[a,b]$ is a pair of sequences $\{t_i\}_{i=0}^n$ and $\{t_i^\star\}_{i=1}^n$ such that*

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b \tag{36.1}$$

and

$$\forall i = 1, \dots, n\colon \quad t_i^\star \in [t_{i-1}, t_i] \tag{36.2}$$

The mesh *of Π is then defined as*

$$\|\Pi\| := \max_{i=1,\dots,n} |t_i - t_{i-1}| \tag{36.3}$$

Given a function $f\colon [a,b] \to \mathbb{R}$,

$$R(f,\Pi) := \sum_{i=1}^n f(t_i^\star)(t_i - t_{i-1}) \tag{36.4}$$

is the Riemann sum *associated with marked partition Π of interval $[a,b]$.*

The quantity $f(t_i^\star)(t_i - t_{i-1})$ represents the area of a rectangle with base $[t_{i-1}, t_i]$ and height $f(t_i^\star)$. (This really applies only if $f(t_i^\star) > 0$; if this value is negative, we get the negative area.) The Riemann sum $R(f,\Pi)$ is then the aggregate area of these rectangles which we can then also think of as the area under the piece-wise constant curve that has height $f(t_i^\star)$ above $(t_{i-1}, t_i]$. We then put forward:

Definition 36.2 (Riemann integrability) *We say that $f\colon [a,b] \to \mathbb{R}$ is* Riemann integrable *if there exists $L \in \mathbb{R}$ such that for each $\epsilon > 0$ there is $\delta > 0$ such that for all marked partitions Π of $[a,b]$,*

$$\|\Pi\| < \delta \;\Rightarrow\; \bigl|R(f,\Pi) - L\bigr| < \epsilon \tag{36.5}$$

The logical proposition "$\forall\epsilon > 0\,\exists\delta > 0\,\forall\Pi =$ marked partition: (36.5) holds" will at times be abbreviated as

$$\lim_{\|\Pi\|\to 0} R(f,\Pi) = L \tag{36.6}$$

while noting that this is not a limit in the sense used earlier (although it can be phrased as convergence of nets). Notwithstanding, exactly the same argument as for limits can be used to show that L above is unique, if it exists:

Lemma 36.3 *If the property in Definition 36.2 holds for L and L', then $L = L'$.*

Proof. Fix $\epsilon > 0$ and let $\delta > 0$ be such that $\|\Pi\| < \delta$ implies $|R(f,\Pi) - L| < \epsilon$. Similarly, let $\delta' > 0$ be such that $\|\Pi\| < \delta'$ implies $|R(f,\Pi) - L| < \epsilon$. Since a marked partition Π exists such that both $\|\Pi\| < \delta$ and $\|\Pi\| < \delta'$, we thus have

$$|L - L'| \leqslant \bigl|R(f,\Pi) - L\bigr| + \bigl|R(f,\Pi) - L'\bigr| < 2\epsilon. \tag{36.7}$$

As this holds for all $\epsilon > 0$, we have $L = L'$. □

The uniqueness justifies introduction of a special symbol:

Definition 36.4 (Riemann integral) *Given reals $a < b$ and a function $f\colon [a,b] \to \mathbb{R}$, the* Riemann integral *of f on interval $[a,b]$ is defined as*

$$\int_a^b f(x)\mathrm{d}x := \lim_{\|\Pi\|\to 0} R(f,\Pi) \tag{36.8}$$

whenever f is Riemann integrable on $[a,b]$.

We will write "f RI" to denote the phrase "f is Riemann integrable" whenever condensed notation is desired. The convention

$$b < a \;\Rightarrow\; \int_b^a f(t)\mathrm{d}t := -\int_a^b f(t)\mathrm{d}t \tag{36.9}$$

is used for convenience (and because it works nicely in manipulations that we will consider later and fits a corresponding property of the Stieltjes integral).

As noted earlier, the above concepts were known already to Newton and Leibnitz who also understood that one has to take the mesh of Π to zero in order to approximate the area under the graph of f better and better. When the notion of a limit became understood precisely, attempts were even made (for instance, by Cauchy who worked with continuous functions) to come up with conditions on f that would guarantee that such approximations converged. Riemann's approach is qualitatively different in that, instead of trying to establish the convergence under increasingly relaxed conditions on the regularity of f, he made integrability a regularity property in its own right.

36.2 Linearity and additivity.

It is easy to check that constant functions are Riemann integrable so the above theory is not vacuous. In order to demonstrate the use of Riemann integrability properly, let us prove some basic properties of the Riemann integral. We start with:

Lemma 36.5 (Linearity) *Let $a < b$ be reals and assume that $f, g\colon [a,b] \to \mathbb{R}$ are Riemann integrable on $[a,b]$. Then for all $\alpha, \beta \in \mathbb{R}$, also $\alpha f + \beta g$ is Riemann integrable on $[a,b]$ and*

$$\int_a^b \bigl(\alpha f(x) + \beta g(x)\bigr)\mathrm{d}x = \alpha \int_a^b f(x)\mathrm{d}x + \beta \int_a^b g(x)\mathrm{d}x \tag{36.10}$$

Proof. Let $\alpha, \beta \in \mathbb{R}$ be fixed and f, g be RI on $[a,b]$. Fix $\epsilon > 0$. Then the RI of f ensures existence of $\delta > 0$ be such that for all marked partitions Π of $[a,b]$,

$$\|\Pi\| < \delta \ \Rightarrow\ \left|R(f,\Pi) - \int_a^b f(x)\mathrm{d}x\right| < \frac{\epsilon}{1 + |\alpha| + |\beta|} \tag{36.11}$$

and let $\delta' > 0$ be such that for all marked partitions Π of $[a,b]$,

$$\|\Pi\| < \delta' \ \Rightarrow\ \left|R(g,\Pi) - \int_a^b g(x)\mathrm{d}x\right| < \frac{\epsilon}{1 + |\alpha| + |\beta|} \tag{36.12}$$

Since the definition (36.4) implies

$$R\bigl(\alpha f + \beta g, \Pi\bigr) = \alpha R(f,\Pi) + \beta R(g,\Pi) \tag{36.13}$$

whenever Π is such that $\|\Pi\| < \min\{\delta, \delta'\}$, the triangle inequality shows

$$\begin{aligned}
&\left|R\bigl(\alpha f + \beta g, \Pi\bigr) - \alpha\int_a^b f(x)\mathrm{d}x - \beta \int_a^b g(x)\mathrm{d}x\right| \\
&\qquad = \left|\alpha\Bigl[R(f,\Pi) - \int_a^b f(x)\mathrm{d}x\Bigr] + \beta\Bigl[R(g,\Pi) - \int_a^b g(x)\mathrm{d}x\Bigr]\right| \\
&\qquad \leqslant |\alpha|\left|R(f,\Pi) - \int_a^b f(x)\mathrm{d}x\right| + |\beta|\left|R(g,\Pi) - \int_a^b g(x)\mathrm{d}x\right| \\
&\qquad < \bigl(|\alpha| + |\beta|\bigr)\frac{\epsilon}{1 + |\alpha| + |\beta|} < \epsilon
\end{aligned} \tag{36.14}$$

As this holds for all $\epsilon > 0$, we have proved that $\alpha f + \beta g$ is RI and

$$\lim_{\|\Pi\| \to 0} R\bigl(\alpha f + \beta g, \Pi\bigr) = \alpha \int_a^b f(x)\mathrm{d}x + \beta \int_a^b g(x)\mathrm{d}x \tag{36.15}$$

This is the desired claim. □

Lemma 36.6 (Monotonicity) *Let $a < b$ be reals and assume that $f, g\colon [a,b] \to \mathbb{R}$ are Riemann integrable on $[a,b]$. Then*

$$\bigl(\forall x \in [a,b]\colon f(x) \leqslant g(x)\bigr) \ \Rightarrow\ \int_a^b f(x)\mathrm{d}x \leqslant \int_a^b g(x)\mathrm{d}x \tag{36.16}$$

Proof. Assuming that $f(x) \geqslant g(x)$ for all $x \in [a,b]$ gives $R(f,\Pi) \leqslant R(g,\Pi)$ for all marked partitions Π of $[a,b]$. The claim then follows from the definition of the integral. □

The content of Lemmas 36.5 and 36.6 can be summarized by saying that

$$\mathcal{R} := \bigl\{f \in \mathbb{R}^{[a,b]}\colon \text{Riemann integrable}\bigr\} \text{ is a vector space} \tag{36.17}$$

and $\varphi\colon \mathcal{R} \to \mathbb{R}$ defined by

$$\varphi(f) := \int_a^b f(x)\mathrm{d}x \tag{36.18}$$

is a *monotone linear functional* on $\mathcal{R}$. We may return to these considerations later when we discuss function spaces more systematically.

Another well-known property of the integral concerns its additivity under subdivision of $[a,b]$. This is the content of:

Lemma 36.7 (Additivity) *Let $a < c < b$ be reals and let $f\colon [a,b] \to \mathbb{R}$. Then*

$$f \text{ RI on } [a,b] \;\Leftrightarrow\; f \text{ RI on } [a,c] \,\wedge\, f \text{ RI on } [c,b] \tag{36.19}$$

and when both (equivalent) statements are TRUE, then

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x \tag{36.20}$$

Our strategy of the proof is to first show $\Rightarrow$ in (36.19) and then prove $\Leftarrow$ along with the formula (36.20). The former requires:

Lemma 36.8 (Cauchy criterion for RI) *Let $f\colon [a,b] \to \mathbb{R}$. Then*

$$f \text{ RI} \;\Leftrightarrow\; \inf_{\delta>0} \sup_{\|\Pi\|,\|\Pi'\|<\delta} \bigl|R(f,\Pi) - R(f,\Pi')\bigr| = 0 \tag{36.21}$$

where Π and Π' on the right denote marked partitions of $[a,b]$.

Leaving the easy proof to homework, we now show:

Proof of $\Rightarrow$ in (36.19). Assume f is RI on $[a,b]$. Pick $\epsilon > 0$ and let $\delta > 0$ be such that the supremum in (36.21) is less than ϵ. Now pick a marked two partitions Π_1 of $[a,c]$ and two marked partitions Π_2 and Π_2' of $[c,b]$, all with mesh less than δ. Write $\Pi := \Pi_1 \circ \Pi_2$ for the concatenation of the partitions Π_1 and Π_2 and $\Pi' := \Pi_1 \circ \Pi_2'$ for the concatenation of the partitions Π_1 and Π_2. It is easy to check that then

$$R(f,\Pi_2) - R(f,\Pi_2') = R(f,\Pi) - R(f,\Pi') \tag{36.22}$$

Since Π and Π' have both mesh less than δ, we thus get

$$\bigl|R(f,\Pi_2) - R(f,\Pi_2')\bigr| = \bigl|R(f,\Pi) - R(f,\Pi')\bigr| < \epsilon \tag{36.23}$$

Proving that f satisfies the Cauchy criterion on $[c,b]$. Hence f is RI on $[c,b]$. A completely analogous argument then shows that f is RI on $[a,c]$ as well. □

In order to address the reverse implication in (36.19), we need:

Lemma 36.9 (Integrability implies boundedness) *Let $f\colon [a,b] \to \mathbb{R}$. Then*

$$f \text{ RI on } [a,b] \;\Rightarrow\; f \text{ bounded on } [a,b] \tag{36.24}$$

In particular, if f is Riemann integrable on $[a,b]$, then

$$\left|\int_a^b f(t)\mathrm{d}t\right| \leqslant \Bigl(\sup_{x\in[a,b]} \bigl|f(x)\bigr|\Bigr)\, |b-a| \tag{36.25}$$

Proof. Suppose f is Riemann integrable. Then there is $\delta > 0$ such that for any marked partition Π with $\|\Pi\| < \delta$, the triangle inequality shows

$$\big|R(f,\Pi)\big| \leqslant \left|\int_a^b f(x)\mathrm{d}x\right| + 1 \tag{36.26}$$

Let $n := \lceil (b-a)/\delta \rceil$ and let $t_i := a + (b-a)i/n$. Given $k = 1,\dots,n$ and $t \in [t_{k-1}, t_k]$, set

$$t_i^\star := \begin{cases} t_{i-1} & \text{if } i \neq k \\ t & \text{if } i = k \end{cases} \tag{36.27}$$

and write $\Pi := (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$. Then

$$f(t)(t_k - t_{k-1}) = R(f,\Pi) - \sum_{i\neq k} f(t_{i-1})(t_i - t_{i-1}) \tag{36.28}$$

Since $\|\Pi\| < \delta$, the triangle inequality along with (36.26) then gives

$$\big|f(t)\big|\,|t_i - t_{i-1}|\big| \leqslant \sum_{i=1}^n \big|f(t_{i-1})\big||t_i - t_{i-1}| + \left|\int_a^b f(x)\mathrm{d}x\right| + 1 \tag{36.29}$$

where we added a term for $i = k$ to make the right-hand side independent of k. Using that $t_i - t_{i-1} = (b-a)/n$, this shows

$$\big|f(t)\big| \leqslant \sum_{i=1}^n \big|f(t_{i-1})\big| + \frac{n}{b-a}\left(\left|\int_a^b f(x)\mathrm{d}x\right| + 1\right) \tag{36.30}$$

But the right hand side does not depend on k or t and so it bounds the supremum of the left-hand side over k and t. The function f is thus bounded as claimed.

As to (36.25), for any marked partition $\Pi := (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ we have

$$\begin{aligned} \big|R(f,\Pi)\big| &\leqslant \sum_{i=1}^n \big|f(t_i^\star)\big|(t_i - t_{i-1}) \\ &\leqslant \Big(\sup_{x\in[a,b]}\big|f(x)\big|\Big)\sum_{i=1}^n (t_i - t_{i-1}) = \Big(\sup_{x\in[a,b]}\big|f(x)\big|\Big)(b-a) \end{aligned} \tag{36.31}$$

Using (36.8), this now extends to the Riemann integral. □

Remark 36.10 The estimate (36.25) tells us that the functional (42.3) is *bounded*, and thus continuous, relative to the supremum norm. We will see later that any sequence $\{f_n\}_{n\in\mathbb{N}}$ of Riemann integrable functions converging in the supremum norm has a Riemann integrable limit, so the space $\mathcal{R}$ is actually complete with respect to this norm.

We are now ready to complete the proof of Lemma 36.7:

Proof $\Leftarrow$ *in* (36.19) *and also* (36.20). Assume f to be RI on $[a,c]$ and $[c,b]$ and fix $\epsilon > 0$. Then there exist $\delta', \delta'' > 0$ such that if Π_1 is a partition of $[a,c]$ with $\|\Pi_1\| < \delta'$ and Π'' is a partition of $[c,b]$ with $\|\Pi''\| < \delta''$ then

$$\left|R(f,\Pi') - \int_a^c f(x)\mathrm{d}x\right| < \epsilon \;\wedge\; \left|R(f,\Pi'') - \int_c^b f(x)\mathrm{d}x\right| < \epsilon \tag{36.32}$$

(The intervals the Riemann sums are over are clear from the partition.)

Let now Π be a marked partition of $[a,b]$ with $\|\Pi\| < \min\{\delta',\delta''\}$. If Π contains c (in the sequence defining partition points), then Π splits into two marked partitions, Π' and Π'' of $[a,c]$ and $[c,b]$, respectively, and we have

$$R(f,\Pi) = R(f,\Pi') + R(f,\Pi''). \tag{36.33}$$

It follows that

$$\begin{aligned}&\left|R(f,\Pi) - \int_a^c f(x)\mathrm{d}x - \int_c^b f(x)\mathrm{d}x\right| \\ &\qquad \leqslant \left|R(f,\Pi') - \int_a^c f(x)\mathrm{d}x\right| + \left|R(f,\Pi'') - \int_c^b f(x)\mathrm{d}x\right| < 2\epsilon\end{aligned} \tag{36.34}$$

The problem is that this bound is restricted to partitions containing c.

Consider now a marked partition $\widetilde{\Pi}$ of $[a,b]$ that does NOT contain c and let $[t_{i-1},t_i]$ be the unique interval in this partition such that $c \in (t_{i-1},t_i)$. Let Π be the partition obtained by adding c to $\widetilde{\Pi}$ and the marked points $c \in [t_{i-1},c] \cap [c,t_i]$. Then

$$\begin{aligned}R(f,\Pi) - R(f,\widetilde{\Pi}) &= f(t_i^\star)(t_i - t_{i-1}) - f(c)(c - t_{i-1}) - f(c)(t_i - c) \\ &= \bigl[f(t_i^\star) - f(c)\bigr](t_i - t_{i-1})\end{aligned} \tag{36.35}$$

and, since by Lemma 36.9 f is bounded on $[a,c]$ and $[c,b]$ and thus on $[a,b]$,

$$\bigl|R(f,\Pi) - R(f,\widetilde{\Pi})\bigr| \leqslant 2\Bigl(\sup_{x\in[a,b]} \bigl|f(x)\bigr|\Bigr)\,\|\Pi\| \tag{36.36}$$

Let δ be such that

$$0 < \delta < \min\{\delta',\delta''\} \ \wedge\ 3\delta\left(\sup_{x\in[a,b]} \bigl|f(x)\bigr|\right) < \epsilon \tag{36.37}$$

If $\|\widetilde{\Pi}\| < \delta$, then (36.34) and (36.36) (along with the fact that $\|\Pi\| \leqslant \|\widetilde{\Pi}\|$) show

$$\begin{aligned}&\left|R(f,\widetilde{\Pi}) - \int_a^c f(x)\mathrm{d}x - \int_c^b f(x)\mathrm{d}x\right| \\ &\qquad \leqslant \bigl|R(f,\Pi) - R(f,\widetilde{\Pi})\bigr| + \left|R(f,\Pi) - \int_a^c f(x)\mathrm{d}x - \int_c^b f(x)\mathrm{d}x\right| \leqslant 3\epsilon\end{aligned} \tag{36.38}$$

This proves the implication $\Rightarrow$ in (36.19) and shows (36.20) as well. □

We note that the fact that Riemann integrability requires boundedness is actually the first sign of problems with the whole concept. Indeed, the function $f(x) := \frac{1}{\sqrt{x}}$ is not Riemann integrable on $[0,1]$ while (as we will show later) it is Riemann integrable on $[a,1]$ for any $a \in (0,1)$ with a well defined limit as $a \to 0^+$. This function can still be included into the theory by using the notion of an *improper integral* but that then unfortunately fails other important properties that the "proper" Riemann integral has.