34. IMPLICIT AND INVERSE-FUNCTION THEOREMS

We will now move to two important results in multivariable differential calculus called the Inverse Function Theorem and the Implicit Function Theorem.

34.1 Inverse Function Theorem.

As seen in the example (33.29), the need for checking that f^{-1} exists and is continuous at the point of interest makes the use of Lemma 33.4 harder than it should, if not close to impossible when the inverse cannot be constructed explicitly. It would be easier if the criteria for differentiability were formulated in terms of f only. This comes in:

Theorem 34.1 (Inverse Function Theorem) *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and $x \in \operatorname{int}(\operatorname{Dom}(f))$ be such that, for some $\delta_0 > 0$ with $B(x,\delta_0) \subseteq \operatorname{Dom}(f)$, the function f is differentiable on $B(x,\delta_0)$ with Df continuous at x and $Df(x)$ invertible. Then there exists $\delta \in (0,\delta_0)$ such that:*

$$f \text{ is injective and open on } B(x,\delta) \tag{34.1}$$

$$f^{-1} \text{ is differentiable on } f(B(x,\delta)) \tag{34.2}$$

and

$$D(f^{-1}) \text{ is continuous at } f(x) \tag{34.3}$$

(Under these circumstances, Lemma 33.4 implies the formula (33.17)*.)*

As we will see in the proof, the key part to prove is (34.1). Once that is established, we get existence of a continuous inverse which reverts us to the setting of the "Inverse function rule" in Lemma 33.4.

Note also that the proof of the single-variable case $n := 1$ has already been discussed before: the non-vanishing of the derivative implies strict monotonicity and thus continuous invertibility etc. Arguments based on monotonicity are not helpful in the multivariable case where we will instead make full use of the linear approximation instead.

For the proof, we first need a lemma:

Lemma 34.2 *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ and let $U \subseteq \operatorname{Dom}(f)$ be open and convex with f differentiable on U and*

$$M := \sup_{x\in U} \|Df(x)\| < \infty \tag{34.4}$$

Then

$$\forall x,y \in U\colon\ \ \|f(y)-f(x)\| \leqslant M\|x-y\| \tag{34.5}$$

In short, a uniform bound on derivative on a convex set implies Lipschitz continuity there.

Proof. The claim is easy to prove if $m = n = 1$; indeed, the Mean-Value Theorem gives us $f(y) - f(x) = f'(t)(y-x)$ for some $t \in (x,y)$ and so the bound on the derivative implies a bound on the ration $|f(y)-f(x)|/|y-x|$. The Mean-Value Theorem fails in higher dimensions, so we proceed by reducing the claim to single-variable setting.

Let $x,y \in U$ and let $\varphi\colon [0,1] \to \mathbb{R}^n$ be defined by $\varphi(t) := (1-t)x + ty$. The convexity of U ensures that $\operatorname{Ran}(\varphi) \subseteq U$. Next define

$$h(t) := \bigl(f(y) - f(x)\bigr) \cdot f \circ \varphi(t) \tag{34.6}$$

Then the fact that

$$h'(t) = \big(f(y) - f(x)\big) \cdot Df\big(\varphi(t)\big)(y-x) \tag{34.7}$$

implies

$$\big|h'(t)\big| \leqslant M\big\|f(y) - f(x)\big\|\|y-x\| \tag{34.8}$$

The Mean-Value Theorem in turn gives us a $t \in [0,1]$ such that $h(1) - h(0) = h'(t)$. Hence we get

$$\big\|f(y) - f(x)\big\|^2 = h(1) - h(0) = h'(t) \leqslant M\big\|f(y) - f(x)\big\|\|y-x\| \tag{34.9}$$

If $\|f(y) - f(x)\| = 0$, then (34.5) holds trivially. In the opposite case, we can divide (39.9) by this factor expression and get (34.5) as well. □

We are ready to begin:

Proof of Theorem 34.1. The proof is based on perturbation arguments and the Banach fixed point theorem. Let x be as in the statement. Set

$$\epsilon := \frac{1}{2\|Df(x)^{-1}\|} \tag{34.10}$$

and use the assumed continuity of Df to find $\delta \in (0, \delta_0)$ such that

$$\forall z \in B(x,\delta)\colon\ \big\|Df(z) - Df(x)\big\| < \epsilon \tag{34.11}$$

Observe that this means that

$$\forall z \in B(x,\delta)\colon\ \big\|Df(x)^{-1}\big\|\big\|Df(z) - Df(x)\big\| \leqslant \frac{1}{2} \tag{34.12}$$

We proceed in three steps:

Step 1: f is injective on $B(x,\delta)$

To prove this, fix $y \in f(B(x,\delta))$ and define $h\colon \mathrm{Dom}(f) \to \mathbb{R}^n$ by

$$h(z) := z + Df(x)^{-1}\big(y - f(z)\big) \tag{34.13}$$

An important fact about this function is that

$$y = f(z) \ \Leftrightarrow\ h(z) = z \tag{34.14}$$

and so for injectivity of f it suffices to show that h has at most one fixed point for each choice of y. To this end we note that

$$Dh(z) = I - Df(x)^{-1}Df(z) = Df(x)^{-1}\big[Df(x) - Df(z)\big] \tag{34.15}$$

and so (34.12) gives

$$\big\|Dh(z)\big\| \leqslant \big\|Df(x)^{-1}\big\|\big\|Df(z) - Df(x)\big\| \leqslant \frac{1}{2} \tag{34.16}$$

Lemma 39.4 along with the fact that $B(x,\delta)$ is open and convex show that

$$\forall z, \tilde{z} \in B(x,\delta)\colon\ \big\|h(z) - h(\tilde{z})\big\| \leqslant \frac{1}{2}\|z - \tilde{z}\| \tag{34.17}$$

But then $h(z) = z$ and $h(\tilde{z}) = \tilde{z}$ imply the inequality $\|z - \tilde{z}\| \leqslant \frac{1}{2}\|z - \tilde{z}\|$ which is only possible if $\|z - \tilde{z}\| = 0$; i.e., $z = \tilde{z}$. Hence, h as at most one fixed point on $B(x,\delta)$ and so, by (34.14), f is injective on $B(x,\delta)$.

Step 2: f is open on $B(x,\delta)$

To prove this let $z_0 \in B(x,\delta)$ and let $r > 0$ be such that

$$\overline{B(z_0,r)} \subseteq B(x,\delta) \tag{34.18}$$

Given any $y \in B(f(z_0),\epsilon r)$, define h as in (34.13). Our goal is to show that there exists $z \in \overline{B(z_0,r)}$ such that $f(z) = y$ which by (34.14) is equivalent to showing that h has a fixed point in $\overline{B(z_0,r)}$. For this we observe that $h(z_0) - z_0 = Df(x)^{-1}(y - f(z_0))$ and so

$$\|h(z_0) - z_0\| \leqslant \|Df(x)^{-1}\|\|y - f(z_0)\| \leqslant \|Df(x)^{-1}\|\epsilon r \leqslant \frac{r}{2} \tag{34.19}$$

Using also (34.17) it follows that, for each $z \in \overline{B(x,r)}$,

$$\|h(z) - z_0\| \leqslant \|h(z) - h(z_0)\| + \|h(z_0) - z_0\| \leqslant \frac{1}{2}\|z - z_0\| + \frac{r}{2} \leqslant r \tag{34.20}$$

proving that h maps $\overline{B(z_0,r)}$ into itself. Since $\overline{B(z_0,r)}$ is a closed subset of the complete metric space $\mathbb{R}^n$ and, by (34.17), h is a Banach contraction, h admits a fixed point in $\overline{B(z_0,r)}$ implying that, with r related to z_0 as above,

$$\forall y \in B\bigl(f(z_0),\epsilon r\bigr)\,\exists z \in \overline{B(z_0,r)}\colon\ f(z) = y \tag{34.21}$$

In light of (34.18) we conclude

$$\forall z_0 \in B(x,\delta)\,\exists r > 0\colon\ B\bigl(f(z_0),\epsilon r\bigr) \subseteq f\bigl(B(x,\delta)\bigr) \tag{34.22}$$

proving that f is open on $B(x,\delta)$ as desired.

Step 3: f^{-1} is differentiable on $f(B(x,\delta))$ and $D(f^{-1})$ is continuous at $f(x)$

From f being injective and open on $B(x,\delta)$ we get existence of a continuous inverse $f^{-1}\colon f(B(x,\delta)) \to B(x,\delta)$ defined on the open set $f(B(x,\delta))$ containing $f(x)$. In light of (34.12), Lemma 33.5 with $A := Df(x)$ and $B := Df(z) - Df(x)$ ensures that $Df(z)^{-1}$ exists for all $z \in B(x,\delta)$ and so, by the "Inverse function rule" in Lemma 33.4, f^{-1} is differentiable on $f(B(x,\delta))$ and (33.17) holds. The latter formula along with Lemma 33.5 also gives that, for all $y \in f(B(x,\delta))$,

$$\begin{aligned}\|D(f^{-1})(y) - D(f^{-1})(f(x))\| &= \|Df(f^{-1}(y)))^{-1} - Df(x)^{-1}\| \\ &\leqslant 2\|Df(x)^{-1}\|^2\|Df(f^{-1}(y)) - Df(x)\|\end{aligned} \tag{34.23}$$

As $y \to f(x)$, the continuity of f^{-1} ensures $f^{-1}(y) \to x$. The continuity of $D(f^{-1})$ at $f(x)$ then follows from the continuity of Df at x. □

A few remarks are in order.

Remark 34.3 First, the continuity of Df at x (needed in the proof of contractivity of h in (34.16–34.17)) is essential. This is seen already in the single-variable case by considering the function

$$f(x) := \begin{cases} x + ax^2\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \tag{34.24}$$

where $a \in \mathbb{R}$ is a parameter. Then $f'(x) = 1 + a\cos(1/x) + 2ax\sin(1/x)$ for $x \neq 0$ and $f'(0) = 1$ and so f' is not continuous at $x = 0$ once $a \neq 0$. For $|a| > 1$, the derivative changes sign and so f is not injective in any open neighborhood of $x = 0$ (though it is injective for $|a| < 1$).

Remark 34.4 Second, the above being said, one can avoid the assumption of continuity of Df at x by replacing it with the assumption that Df exists and is invertible in an open neighborhood of x. The resulting "Everywhere differentiable Inverse Function Theorem" is harder to prove (as the Banach fixed point theorem cannot be applied) but can be phrased using tools already introduced here. See Terry Tao's blog on this result.

Remark 34.5 Third, although Theorem 34.1 gives us existence of a local inverse, nothing can be said about global invertibility when $n \geqslant 2$ — not even under the assumption of connectivity of $\text{Dom}(f)$ and global invertibility of Df which suffices for this purpose when $n = 1$ by the Mean-Value Theorem. To see the difference between $n = 1$ and $n \geqslant 2$, consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) := \begin{pmatrix} e^x \cos(y) \\ e^x \sin(y) \end{pmatrix} \tag{34.25}$$

Then Df is given by the Jacobian matrix

$$Df(x,y) = \begin{pmatrix} e^x \cos(y) & -e^x \sin(y) \\ e^x \sin(y) & e^x \cos(y) \end{pmatrix} \tag{34.26}$$

whose determinant is $\det(Df)(x,y) = e^{2x}$ which is everywhere non-zero. Since Df is also continuous, f is locally invertible yet, being periodic in the y-direction, definitely not injective on all of $\mathbb{R}^2$.

Remark 34.6 Fourth, the Inverse Function Theorem is sometimes stated under the assumption that the differential Df is continuous on an open neighborhood of x; then also $D(f^{-1})$ is continuous on an open neighborhood of $f(x)$. This is checked via (39.14).

34.2 Implicit Function Theorem.

Although it may not be apparent from the proof, the principal idea driving the Inverse Function Theorem is linear approximation: Indeed, with $Df(x)$ well defined, f is well approximated near x by the linear function

$$g(z) := f(x) + Df(x)(z - x) \tag{34.27}$$

Assuming $Df(x)$ is also invertible, then $g(z) = y$ can be solved by

$$z = x + Df(x)^{-1}(y - f(x)) \tag{34.28}$$

and so (assuming also continuity of Df at x), the function f is locally invertible near x if its linear approximation is invertible. (Recall that, by Remark 34.4, having Df invertible in a neighborhood of x in fact suffices for the conclusion.)

We will now use the same idea to resolve the following problem that arises quite often in practice. Suppose we are given m functions $F_1, \dots, F_m\colon \mathbb{R}^{n+m} \to \mathbb{R}$ of $n + m$-variables. Writing these functions as $F_1(x,y), \dots, F_m(x,y)$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, we would like to solve for y from the equations

$$\begin{aligned} F_1(x,y) &= 0 \\ &\vdots \\ F_m(x,y) &= 0 \end{aligned} \tag{34.29}$$

That such an m-tuple of functions $y_1(x),\dots,y_m(x)$ might exists can perhaps be argued by elimination: First express y_m from the first equation as a function of the variables $x_1,\dots,x_n,y_1,\dots,y_{m-1}$, then plug this into the second equation and express y_{m-1} as a function of $x_1,\dots,x_n,y_1,\dots,y_{m-2}$ from the second equation, etc. However, this ignores that we are in no way guaranteed that such elimination procedure can be performed to the end in the given order and, in fact, in any order.

It is not hard to convince oneself that to express the y's simultaneously from the equations, all we need is that the map

$$y \mapsto F(x,y) := (F_1(x,y),\dots,F_m(x,y)) \tag{34.30}$$

is invertible for each x near the "point" $0 \in \mathbb{R}^m$. This is exactly what we addressed in the Inverse Function Theorem. Of course, we actually need at least one x, to be called x_0 below, for which the map (34.30) evaluates to 0 at some some y_0. Since the Inverse Function Theorem gives us only a local inverse, also the construction of the solutions will be just local. This is pretty much all that is going on in:

Theorem 34.7 (Implicit Function Theorem) *Given naturals $n,m \geqslant 1$, let $F\colon \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a function and $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^m$ points such that $(x_0,y_0) \in \operatorname{int}(\operatorname{Dom}(F))$ and*

$$F(x_0,y_0) = 0 \tag{34.31}$$

Assume that, for some $\delta_0 > 0$, the function F is differentiable on $B((x_0,y_0),\delta_0)$ with DF continuous at (x_0,y_0) and its y-minor

$$D_yF(x,y) := \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(x,y) & \dots & \frac{\partial F_1}{\partial y_m}(x,y) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1}(x,y) & \dots & \frac{\partial F_m}{\partial y_m}(x,y) \end{pmatrix} \tag{34.32}$$

invertible at $(x,y) = (x_0,y_0)$. Then there exists $\delta > 0$ and a continuous function $g\colon \mathbb{R}^n \to \mathbb{R}^m$ with $\operatorname{Dom}(g) = B(x_0,\delta)$ *such that*

$$g(x_0) = y_0 \tag{34.33}$$

and

$$\forall x \in B(x_0,\delta)\colon\ F\big(x,g(x)\big) = 0 \tag{34.34}$$

This g is differentiable on $B(x,\delta)$ with Dg continuous at x_0. Moreover, any two functions g that obey (34.33–34.34) and are continuous at x_0 coincide in an open ball centered at x_0.

Proof. Define $f\colon \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ with $\operatorname{Dom}(f) = B((x_0,y_0),\delta_0)$ by

$$f(x,y) := \begin{pmatrix} x \\ F(x,y) \end{pmatrix} \tag{34.35}$$

where the two entries represent column vectors of dimension n and m, respectively. Using the same convention, the differential of f takes the block-matrix form

$$Df(x,y) = \begin{pmatrix} I & 0 \\ D_xF(x,y) & D_yF(x,y) \end{pmatrix} \tag{34.36}$$

where D_xF is the $m \times n$ Jacobian matrix of partial derivatives of the m-components of F with respect to the n components of x.

Next we will show that that Df is continuous at (x_0, y_0). For this we note that, for all $(x,y) \in B((x_0,y_0),\delta')$,

$$Df(x,y) - Df(x_0,y_0) = \begin{pmatrix} 0 & 0 \\ D_xF(x,y) - D_xF(x,y) & D_yF(x,y) - D_yF(x_0,y_0) \end{pmatrix} \tag{34.37}$$

Multiplying this matrix by a vector (a,b), where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, results in a vector whose first n coordinates vanish and the remaining m coordinates are those of the vector

$$\big(D_xF(x,y) - D_xF(x,y)\big)a + \big(D_yF(x,y) - D_yF(x_0,y_0)\big)b \tag{34.38}$$

This vector is directly identified with

$$\big(DF(x,y) - DF(x_0,y_0)\big)\begin{pmatrix} a \\ b \end{pmatrix} \tag{34.39}$$

and so optimizing over (a,b) we get

$$\big\|Df(x,y) - Df(x_0,y_0)\big\| = \big\|DF(x,y) - DF(x_0,y_0)\big\| \tag{34.40}$$

proving that Df is continuous at (x_0,y_0). Having settled continuity, we finally note that the inverse of the block matrix in (34.37) can be guessed to be

$$Df(x_0,y_0)^{-1} = \begin{pmatrix} I & 0 \\ -(D_yF)^{-1}D_xF & (D_yF)^{-1} \end{pmatrix} \tag{34.41}$$

where the matrices are evaluated at (x_0,y_0). In particular, Df is invertible at (x_0,y_0).

The above validates the conditions of the Inverse Function Theorem which states the following: There exists $\delta' > 0$ such that f is invertible on $f(B')$ for $B' := B((x_0,y_0),\delta')$ with the inverse f^{-1} differentiable on $f(B')$ with $D(f^{-1})$ continuous at $f(x_0,y_0)$. In light of (34.35) and the fact that $f \circ f^{-1}$ is the identity on $f(B')$, the inverse f^{-1} necessarily takes the form

$$f^{-1}(u,v) = \begin{pmatrix} u \\ h(u,v) \end{pmatrix} \tag{34.42}$$

for some $h\colon \mathbb{R}^{n+m} \to \mathbb{R}$ defined on $\operatorname{Dom}(h) := f(B')$ such that

$$\forall (u,v) \in B'\colon\ F\big(u, h(u,v)\big) = v \tag{34.43}$$

Now observe that $(x_0,y_0) \in B'$ implies that $f(x_0,y_0) = (x_0,0) \in B'$. The fact that B' is a Euclidean ball containing the point $(x_0,0)$ implies that, for some $\delta > 0$, the Euclidean ball $B(x_0,\delta)$ in $\mathbb{R}^n$ obeys $B(x_0,\delta) \times \{0\} \subseteq B'$. For each $x \in B(x_0,\delta)$ we then define

$$g(x) := h(x,0) \tag{34.44}$$

and observe that, since $f^{-1}(x_0,0) = (x_0,y_0)$, we have

$$g(x_0) = h(x_0,0) = y_0 \tag{34.45}$$

and so g indeed obeys (34.33–34.34).

The differentiability of f^{-1} implies differentiability of h and thus of g as well, and the continuity of $D(f^{-1})$ at $(x_0, 0)$ implies continuity of Dg at x_0. To prove uniqueness, suppose that g and $\tilde{g}$ are two continuous functions satisfying (34.33–34.34). Let $\delta'' \in (0, \delta]$ be such that $x \in B(x_0, \delta'')$ implies $(x, g(x)) \in B((x_0, y_0), \delta')$ and $(x, g(x)) \in B((x_0, y_0), \delta')$. (This follows by continuity of g and $\tilde{g}$ at x_0.) Hence we get

$$\forall x \in B(x_0, \delta'')\colon\;\; f(x, g(x)) = f(x, \tilde{g}(x)) \tag{34.46}$$

But f is injective on $B((x_0, y_0), \delta')$ and so $\tilde{g}(x) = g(x)$ for all $x \in B(x_0, \delta'')$. □

34.3 Examples.

In order to demonstrate the use of the Implicit Function Theorem, suppose we wish to solve for u and v from the equations

$$\begin{aligned} xu - yv &= 0 \\ yu + xv &= 1 \end{aligned} \tag{34.47}$$

That a solution exist is checked by plugging $x = 1 = v$ and $y = 0 = u$. We can write this as $F(x, y, u, v) = 0$ for $F\colon \mathbb{R}^4 \to \mathbb{R}^2$ given as

$$F(x, y, u, v) := \begin{pmatrix} xu - yv \\ yu + xv - 1 \end{pmatrix} \tag{34.48}$$

The Implicit Function Theorem asks us to compute the derivatives of F with respect to u and v. This yields

$$D_{(u,v)} F(x, y, u, v) := \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \tag{34.49}$$

The determinant of this matrix equals $x^2 + y^2$ which is positive, and the matrix is thus invertible, away from $(x, y) = (0, 0)$ for which a solution cannot exist anyways. The Implicit Function Theorem then gives us existence of continuously differentiable functions $u = u(x, y)$ and $v = v(x, y)$ with domain $\mathbb{R}^2 \smallsetminus \{(0, 0)\}$ such that

$$\forall (x, y) \in \mathbb{R}^2 \smallsetminus \{(0, 0)\}\colon\;\; F\big(x, y, u(x, y), v(x, y)\big) = 0 \tag{34.50}$$

Thanks to connectivity of the domain, and the local uniqueness, these functions are unique. (All of this can be of course proved directly by solving for u and v from (34.47) but this is not the point here.)

With $u = u(x, y)$ and $v = v(x, y)$ defined and shown to be differentiable, we can ask for the value of their derivatives at a particular point of the solution. This is answered by methods of calculus. Indeed, differentiating the equations (34.47) with respect to x after $u = u(x, y)$ and $v = v(x, y)$ have been plugged in yields

$$\begin{aligned} x \frac{\partial u}{\partial x} - y \frac{\partial v}{\partial x} &= -u \\ y \frac{\partial u}{\partial x} + x \frac{\partial v}{\partial x} &= -v \end{aligned} \tag{34.51}$$

and similarly for the derivative with respect to y.

It is no coincidence that the coefficient matrix associated with this system of linear equations is exactly that in (34.49). Indeed, with g differentiable, we can differentiate the equation $F(x,g(x)) = 0$ via the Chain Rule to get

$$D_xF\bigl(x,g(x)\bigr) + D_yF\bigl(x,g(x)\bigr)Dg(x) = 0 \tag{34.52}$$

from which we compute

$$Dg(x) = -D_yF\bigl(x,g(x)\bigr)^{-1}D_xF\bigl(x,g(x)\bigr) \tag{34.53}$$

where the existence of the inverse is effectively shown inside the proof of the Inverse Function Theorem. The process of differentiating functions that are obtained (often inexplicitly) as solutions to equations is referred to as *implicit differentiation*.

The mathematically hard part of implicit differentiation technique is to guarantee the existence of the needed derivatives; the rest is just a straightforward application of the Chain Rule and some linear algebra. We already had one example on this earlier but to demonstrate this technique further, suppose that we are asked to compute y' and z' for functions $y = y(x)$ and $z = z(x)$ that solve the equations

$$\begin{aligned} f(x,y,z) &= 0 \\ g(x,y,z) &= 0 \end{aligned} \tag{34.54}$$

where $f,g\colon \mathbb{R}^3 \to \mathbb{R}$ are continuously differentiable functions. The Implicit Function Theorem tells us that these functions exist at least locally locally near the points where

$$\begin{pmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix} \tag{34.55}$$

is invertible. Differentiating the equations (34.54) with respect to x — as done generally and in formal notation in (34.52) — then gives us

$$\begin{aligned} \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' &= 0 \\ \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}y' + \frac{\partial g}{\partial z}z' &= 0 \end{aligned} \tag{34.56}$$

where we suppress the arguments of the functions for brevity. Thanks to the invertibility of the matrix (34.55), hence we get

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = -\begin{pmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial x} \end{pmatrix} \tag{34.57}$$

where all functions on the right are evaluated at the point $(x,y(x),z(x))$.