30. Mean-Value Theorems

Equipped with the concept of the derivative and, in particular, the characterization of local extrema by the first derivative test, we will now draw a couple of interesting conclusions. The key word of this chapter is the Mean-Value Theorem.

30.1 Mean-Value Theorems.

We start with with a theorem that goes back to M. Rolle in 1691 albeit with a rigorous proof first given by A.L. Cauchy in 1823. The name Mean-Value Theorem is usually use to refer to the version attributed to J.-L. Lagrange.

Theorem 30.1 (Mean-Value Theorems of Rolle, Lagrange and Cauchy) *Let $a < b$ be reals and $f\colon [a,b] \to \mathbb{R}$ a function (with $\operatorname{Dom}(f) = [a,b]$) that is continuous on $[a,b]$ and differentiable on (a,b). Then*

(1) (Rolle's Theorem)

$$f(a) = f(b) \;\Rightarrow\; \exists x \in (a,b)\colon\; f'(x) = 0 \tag{30.1}$$

(2) (Lagrange's Theorem)

$$\exists x \in (a,b)\colon\; f'(x) = \frac{f(b)-f(a)}{b-a} \tag{30.2}$$

(3) (Cauchy's Theorem) *If also $g\colon [a,b] \to \mathbb{R}$ (with $\operatorname{Dom}(f) = [a,b]$) is continuous on $[a,b]$ and differentiable on (a,b), then*

$$\forall x \in (a,b)\colon g'(x) \neq 0 \tag{30.3}$$

implies

$$g(b) \neq g(a) \;\wedge\; \exists x \in (a,b)\colon \frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)} \tag{30.4}$$

Proof. We start with (1). Suppose f is as given with $f(a) = f(b)$. Then one of the three alternatives occur

$$\sup_{x\in[a,b]} f(x) > f(a) \;\vee\; \inf_{x\in[a,b]} f(x) < f(a) \;\vee\; \forall x \in [a,b]\colon\; f(x) = f(a) \tag{30.5}$$

Since (by Corollary 24.17) a continuous real-valued function on a compact set achieves its minimum and maximum, in all three cases f has a global extremum (maximum or minimum) at some $x \in (a,b)$. By Theorem 29.11, we then have $f'(x) = 0$ as desired.

Moving to the proof of (2), let $h\colon [a,b] \to \mathbb{R}$ be defined by

$$h(x) := f(x) - \frac{f(b)-f(a)}{b-a}(x-a) \tag{30.6}$$

Then, as is readily checked, h is continuous on $[a,b]$ and differentiable on (a,b) (being the difference of two functions with these properties). Moreover, $h(a) = f(a) = h(b)$ and so, by (1), there exists $x \in (a,b)$ with

$$0 = h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}. \tag{30.7}$$

This is the statement in (30.2).

Part (3), which subsumes part (2), is proved using a similar trick. First note that (30.2) and (30.3) imply $g(a) \neq g(b)$ for otherwise there would a point $x \in (a,b)$ where $g'(x)$ vanishes. This proves the first half of (30.4) and allows us to define $h\colon [a,b] \to \mathbb{R}$ by

$$h(x) := f(x) - \frac{f(b)-f(a)}{g(b)-g(a)}\big(g(x)-g(x)\big) \tag{30.8}$$

which is then again continuous on $[a,b]$ and differentiable on (a,b). By (1) we thus get the existence of $x \in (a,b)$ such that

$$0 = h'(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)}g'(x) \tag{30.9}$$

Dividing by $g'(x)$, which is non-zero by (30.3), we get the second half of (30.4). □

Lagrange's MVT can be interpreted by saying that, for each continuous differentiable function on interval $[a,b]$ there is a point where the tangent line has the same slope as the secant line between the endpoints of the interval.

Another "practical" consequence of the theorem is that, for a body that moves distance L in time T, there must be a time $t \in (0,T)$ where the instantaneous speed of motion equals L/T. However, this "reasoning" seems to rely on the speed changing continuously in time. This is not needed for Lagrange's MVT to hold, but something close enough to that saves the day; see Theorem 30.6.

30.2 Applications.

Moving back to mathematics, we will now go over a couple of standard applications of Mean-Value Theorems. The first one concerns a well-known characterization of monotone differentiable functions:

Lemma 30.2 *Let $a < b$ be reals and $f\colon [a,b] \to \mathbb{R}$ a function (with* $\mathrm{Dom}(f) = [a,b]$*) that is continuous on $[a,b]$ and differentiable on (a,b). Then*

$$f \text{ non-decreasing on } [a,b] \;\Leftrightarrow\; \forall x \in (a,b)\colon\; f'(x) \geqslant 0 \tag{30.10}$$

Proof. We start with the easy direction ⇒. Indeed, assume that f is as above and non-decreasing. Let $x, y \in (a,b)$ be such that $y \neq x$. Then

$$\frac{f(y)-f(x)}{y-x} \geqslant 0 \tag{30.11}$$

and so $f'(x) \geqslant 0$ by the definition of the derivative (29.1).

The proof of ⇔ is done by contrapositive. Indeed, assume that $x, y \in [a,b]$ are such that $x < y$ and $f(y) < f(x)$. Then (30.3) implies existence of $z \in (x,y) \subseteq (a,b)$ such that

$$f'(z) = \frac{f(y)-f(x)}{y-x} < 0. \tag{30.12}$$

Hence, non-negativity of f' on (a,b) forces upward monotonicity of f on $[a,b]$. □

The use of non-strict monotonicity and non-negativity of the derivative is necessary. This is because a strictly increasing differentiable functions may not always have a strictly positive derivative. (The function $f(x) = x^3$ is an example.) The above theorem has a version that is useful in applications:

Corollary 30.3 *Let $f,g\colon[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) and such that*

$$f(a)\leqslant g(a)\ \wedge\ \forall x\in(a,b)\colon f'(x)\leqslant g'(x) \tag{30.13}$$

Then

$$\forall x\in[a,b]\colon\ f(x)\leqslant g(x) \tag{30.14}$$

Proof. Let $h(x):=g(x)-f(x)$. By Lemma 30.2, h is non-decreasing. Since $h(a)\geqslant 0$ we have $h(x)\geqslant 0$ for all $x\in[a,b]$. □

One application arises when we need to prove a bound on a function. An example of this is $f(x):=\sin(x)$ (which we can think of as the unique solution to the second-order ODE $f''=-f$ with $f(0)=0$ and $f'(0)=1$). Then $f'(x)=\cos(x)$ and so once we know that $\cos(x)\leqslant 1$, the above gives $f(x)\leqslant x$ for all $x\geqslant 0$.

Another application is to the solutions of ordinary differential equations. Here is a statement in this vain:

Lemma 30.4 (Comparison of ODEs) *Let $F,G\colon\mathbb{R}\to\mathbb{R}$ (with domain all of the reals) be continuous functions with*

$$\forall u,v\in\mathbb{R}\colon\ u\leqslant v\ \Rightarrow\ F(u)\leqslant G(v). \tag{30.15}$$

Let $y,z\colon[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) functions that solve the ordinary differential equations

$$\forall x\in(a,b)\colon\ y'(x)=F\bigl(y(x)\bigr)\ \wedge\ z'(x)=G\bigl(z(x)\bigr) \tag{30.16}$$

with the "initial" values such that $y(a)<z(a)$. Then

$$\forall x\in[a,b]\colon\ y(x)\leqslant z(x) \tag{30.17}$$

We leave the easy proof of this lemma to the reader. To demonstrate this on an example, consider the ODE

$$y'=y+\sqrt{y} \tag{30.18}$$

with initial value $y(0)=1$. Then $y'\geqslant y$ and so the above shows that y is bounded from below by the solution to the ODE

$$z'=z \tag{30.19}$$

with initial value $z(0)=a$ for any $a<1$. (This ODE happens to be solved by $z(x)=a\mathrm{e}^x$ so, taking $a\to 1$ from below we get $y(x)\geqslant\mathrm{e}^x$ for all $x\geqslant 0$.) These ideas drive the technique for solving *differential inequalities* which sometimes arise in applications.

30.3 Intermediate-value property of the derivative.

Our next application of the Mean-Value Theorems, or rather the first-derivative test that underlies its proof, is of a somewhat abstract nature. We start with a definition:

Definition 30.5 *Let $I\subseteq\mathbb{R}$ be a non-degenerate interval. A function $f\colon\mathbb{R}\to\mathbb{R}$ with $I\subseteq\mathrm{Dom}(f)$ is said to have the* intermediate value property *(IVP) on I if*

$$\forall x,y\in I\colon\ x<y\ \Rightarrow\ f\bigl([x,y]\bigr)\text{ is an interval} \tag{30.20}$$

We wrote the above definition an a somewhat succinct form. There are equivalent alternative forms of this definition; for instance,

$$\forall x, y \in I \colon x \leqslant y \,\wedge\, f(x) \leqslant f(y) \;\Rightarrow\; \forall t \in \big[f(x), f(y)\big]\, \exists z \in [x, y] \colon f(z) = t \tag{30.21}$$

which is checked to be equivalent to the above using the fact that intervals are (the only) connected subsets of the reals.

Notably, at some point in early 19th century, the IVP was considered a possible candidate for the concept of "continuity." (Recall that the Intermediate Value Theorem shows that any continuous function on (a, b) will have an IVP.) What is perhaps more relevant is that there are functions that have an IVP yet are not continuous. (Examples will be given after the next result.)

Theorem 30.6 (Darboux' theorem) *Let $f \colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then f' has the intermediate value property on (a, b).*

Proof. Let $x, y \in (a, b)$ be such that $x < y$ and, without loss of generality, $f'(x) < f'(y)$ (otherwise, swap $-f$ for f). Pick $t \in (f'(x), f'(y))$ and define $h(u) := f(u) - tu$. Then $h'(x) = f'(x) - t < 0$ and so $h(x)$ is not a local minimum of h on $[x, y]$. Similarly, $h'(y) = f'(y) - t > 0$ and so $h(y)$ is not a local minimum of h on $[x, y]$ either. As h is continuous, Corollary 24.17 implies that it achieves its minimum in (x, y) and so, by Theorem 29.11, there exists $u \in (x, y)$ such that $h'(u) = 0$. This translates into $f'(u) = t$. As t was arbitrary in $(f'(x), f'(y))$, the function f' has the IVP on (a, b). □

Hereby we get:

Corollary 30.7 *Let $f \colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then f' has no discontinuities of the first kind on (a, b).*

Proof. As is readily checked, if a function h has a discontinuity of the first kind at $x \in \operatorname{int}(\operatorname{Dom}(h))$, then h fails to have IVP in an open interval containing x. □

To give some examples, note that the function in (27.23) cannot be a derivative because it has a discontinuity of the first kind at every rational (one such discontinuity suffices). For a positive example, consider the function $f \colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \tag{30.22}$$

where, $\sin(\cdot)$ and $\cos(\cdot)$ are the sine and cosine functions the reader knows (at least intuitively) from trigonometry. (We will define these using analytic tools later.) Then f is continuous and differentiable at each $x \neq 0$ with

$$f'(x) = 2x \sin(1/x) - \cos(1/x) \tag{30.23}$$

This means that $\lim_{x \to 0} f'(x)$ does NOT exist. Yet

$$\frac{f(x) - f(0)}{x - 0} = x \sin(1/x) \tag{30.24}$$

shows that $f'(0) = 0$. So f' exists on all of $\mathbb{R}$, has the IVP but is NOT continuous at 0. We will see that this example can be boosted to have a function which is differentiable on $\mathbb{R}$ but whose derivative is not continuous at any rational.