29. THE DERIVATIVE

One reason for treating limits is the definition of a concept of the derivative that is foundational for Calculus and many aspects of Analysis.

29.1 Definition.

In Calculus, the definition of the derivative is motivated by studying the "slope" of secant lines to the graph of a function f at x and using these to determine the asymptotic "slope" of the tangent line to the graph at x. We will not go into these ramifications here; instead we simply present the result:

Definition 29.1 (Derivative) *Let $f: \mathbb{R} \to \mathbb{R}$ and $x \in \operatorname{int}(\operatorname{Dom}(f))$. We say that f has* derivative at x *or* is differentiable at x *if the limit*

$$f'(x) := \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \tag{29.1}$$

exists. We then call $f'(x)$ the derivative of f at x.

The notation $f'(x)$, which suggests that we can treat the derivative of f as a function, is referred to as the *Lagrange notation*. The alternative (and widely used) *Leibnitz notation* $\frac{\mathrm{d}f}{\mathrm{d}x}$ does not have that feature but it captures better the definition of the object as the ratio of increment of f and the increment of x.

A common way to write the limit in (29.1) is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{29.2}$$

Note that in both cases the expression under the limit is not defined when $z = x$ in (29.1) or $h = 0$ in (29.2). This is no loss as the value at x is irrelevant for the limit at x.

The definitions (29.1) and (29.2) make sense whenever $x \in \operatorname{Dom}(f)$ is NOT isolated, but we we usually require that at least one of the intervals $[x, x+\delta)$ or $(x-\delta, x]$ belong to $\operatorname{Dom}(f)$. Taking this minimal assumption, we put forward:

Definition 29.2 *Let $f: \mathbb{R} \to \mathbb{R}$ and let $x \in \operatorname{Dom}(f)$. If there exists $\delta > 0$ such that $[x, x+\delta) \subseteq \operatorname{Dom}(f)$, we define the* right derivative of f at x *by*

$$\frac{\mathrm{d}f}{\mathrm{d}x^+}(x) = f'^{,+}(x) := \lim_{z \to x^+} \frac{f(z) - f(x)}{z - x} \tag{29.3}$$

and, if the limit exists, say that f is right-differentiable at x. *Similarly, if in turn there exists $\delta > 0$ such that $(x-\delta, x] \subseteq \operatorname{Dom}(f)$, we define the* left derivative of f at x *by*

$$\frac{\mathrm{d}f}{\mathrm{d}x^-}(x) = f'^{,-}(x) := \lim_{z \to x^-} \frac{f(z) - f(x)}{z - x} \tag{29.4}$$

and, if the limit exists, say that f is left-differentiable at x.

We leave it to the reader to prove:

Lemma 29.3 *If $f: \mathbb{R} \to \mathbb{R}$ is such that $x \in \operatorname{int}(\operatorname{Dom}(f))$, then*

$$f'(x) \text{ exists } \Leftrightarrow f'^{,+}(x), f'^{,-}(x) \text{ exist } \wedge f'^{,+}(x) = f'^{,-}(x) \tag{29.5}$$

If both sides are TRUE, then $f'(x) = f'^{,+}(x) = f'^{,-}(x)$.

For a number of elementary functions the computation of the derivative is rather straightforward. For instance, if f is constant then $f(z) - f(x) = 0$ for all z and so $f'(x) = 0$. Or if f is linear, i.e., $f(z) := az + b$, then $f(z) - f(x) = a(z - x)$ and $f'(x) = a$. Only slightly more difficult is the case of $f(z) := z^2$ where we use that

$$\frac{f(z) - f(x)}{z - x} = \frac{z^2 - x^2}{z - x} = z + x \tag{29.6}$$

and so $f'(x) = 2x$. Similarly, for $f(z) := z^n$ with $n \geqslant 1$ natural we obtain

$$\frac{f(z) - f(x)}{z - x} = \frac{z^n - x^n}{z - x} = \sum_{k=0}^{n-1} z^k x^{n-k-1} \tag{29.7}$$

which yields $f'(x) = nx^n$. The latter argument generalizes to negative powers; indeed, taking $f(x) := x^{-n}$, for $x, z \neq 0$ we have

$$\frac{f(z) - f(x)}{z - x} = \frac{z^{-n} - x^{-n}}{z - x} = -\frac{1}{xz}\,\frac{z^{-n} - x^{-n}}{z^{-1} - x^{-1}} = -\frac{1}{xz}\sum_{k=0}^{n-1} z^{-k} x^{k-n+1} \tag{29.8}$$

which gives $f'(x) = -nx^{-n-1}$. All of these calculations become special instances of:

Lemma 29.4 (Power rule) *Given $\alpha \in \mathbb{R}$, let $f\colon \mathbb{R}^+ \to \mathbb{R}$ be defined by $f(x) := x^\alpha$. Then*

$$\forall x > 0\colon\ f'(x) = \alpha x^{\alpha-1} \tag{29.9}$$

For $\alpha \geqslant 1$, we also get $f'^{,+}(0) = 0$.

Proof for $\alpha \in \mathbb{Q}$. The above computations proved the result for $\alpha \in \mathbb{Z}$. Writing $\alpha = p/q$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N} \smallsetminus \{0\}$, we now observe that

$$\frac{z^{p/q} - x^{p/q}}{z - x} = \frac{z^{p/q} - x^{p/q}}{z^{1/q} - x^{1/q}} \Bigg/ \frac{z - x}{z^{1/q} - x^{1/q}} \tag{29.10}$$

Using (29.7–29.8) with the help of the continuity of the q-th root function, we get

$$\lim_{z \to x} \frac{z^{p/q} - x^{p/q}}{z^{1/q} - x^{1/q}} = \lim_{z \to x} \frac{(z^{1/q})^p - (x^{1/q})^p}{z^{1/q} - x^{1/q}} = p(x^{1/q})^{p-1} \tag{29.11}$$

while for the second ratio we similarly get

$$\lim_{z \to x} \frac{z - x}{z^{1/q} - x^{1/q}} = \lim_{z \to x} \frac{(z^{1/q})^q - (x^{1/q})^q}{z^{1/q} - x^{1/q}} = q(x^{1/q})^{q-1} \tag{29.12}$$

The statement follows from the quotient rule for the limits and elementary algebra. □

The proof for irrational α cannot be done solely via the above algebraic rewrites because these powers are defined by a limit. In this case it is easier to proceed by a rewrite via the exponential and logarithmic function.

29.2 Linear approximation.

Being differentiable should be understood as a statement of regularity of the underlying function. This regularity is stronger than continuity:

Lemma 29.5 *Let $f\colon \mathbb{R}\to\mathbb{R}$ and $x\in\text{int}(\text{Dom}(f))$. Then*

$$f'(x) \text{ exists } \Rightarrow\ f \text{ is continuous at } x \tag{29.13}$$

Proof. The proof will use an idea that we will recycle a few times over later. Assume $f'(x)$ exists and define $u_x\colon \mathbb{R}\to\mathbb{R}$ (with $\text{Dom}(u_x) := \text{Dom}(f)$) by

$$u_x(z) := \begin{cases} \dfrac{f(z)-f(x)}{z-x} - f'(x), & \text{if } z\neq x,\\ 0, & \text{if } z = x.\end{cases} \tag{29.14}$$

Then

$$f(z)-f(x) = \bigl[f'(x)+u_x(z)\bigr](z-x) \tag{29.15}$$

The existence of $f'(x)$ translates into

$$\lim_{z\to x} u_x(z) = 0 \tag{29.16}$$

and so there exists $\delta_0>0$ such that $|z-x|<\delta_0$ implies $|u_x(z)|\leqslant 1$. But then

$$|z-x|<\delta_0\ \Rightarrow\ \bigl|f(z)-f(x)\bigr| \leqslant \bigl(1+|f'(x)|\bigr)|x-z| \tag{29.17}$$

and so, given $\epsilon>0$, setting $\delta := \min\{\delta_0, \epsilon/(1+|f'(x)|)\}$ we get

$$|z-x|<\delta\ \Rightarrow\ \bigl|f(z)-f(x)\bigr|<\epsilon \tag{29.18}$$

proving continuity of f at x. □

The converse to the implication in (29.13) fails; indeed, there exist continuous functions $\mathbb{R}\to\mathbb{R}$ that are not differentiable at any point.

As discussed at length in Calculus, the existence of a derivative at x is actually equivalent to f admitting a linear approximation near x. This is summarized in:

Lemma 29.6 (Linear approximation) *Let $f\colon\mathbb{R}\to\mathbb{R}$ and $x\in\text{int}(\text{Dom}(f))$. Then*

$$f'(x) \text{ exists } \Leftrightarrow\ \exists a\in\mathbb{R}\colon\ \inf_{\delta>0}\ \sup_{\substack{z\in\text{Dom}(f)\\ |z-x|<\delta}} \frac{1}{\delta}\bigl|f(z)-f(x)-a(z-x)\bigr| = 0 \tag{29.19}$$

The infimum can be replaced by the limit as $\delta\to 0^+$.

We leave the proof of this lemma to homework while noting that the representation using (29.14) is likely to be useful.

29.3 "Rules" for derivatives.

While the derivative can be computed for a number of elementary functions, general manipulations with the derivative are made considerably easier thanks to the various "rules" being available. These are generally inherited from the corresponding "rules" for limits. We start with:

Lemma 29.7 (Sum and Product Rule) *Let $f,g\colon\mathbb{R}\to\mathbb{R}$ and $x\in\text{int}(\text{Dom}(f))$. If f and g are both differentiable at x then so are $f+g$ and $f\cdot g$ and*

$$(f+g)'(x) = f'(x)+g'(x) \tag{29.20}$$

and

$$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x) \tag{29.21}$$

hold.

Proof. As to the sum, here we note that, for each $z \neq x$,

$$\frac{(f+g)(z) - (f+g)(x)}{z-x} = \frac{f(z) - f(x)}{z-x} + \frac{g(z) - g(x)}{z-x} \tag{29.22}$$

Hereby we get (29.20) using the Sum Rule for limits; cf Lemma 27.6.

Concerning the product, here we need the rewrite

$$\frac{(f \cdot g)(z) - (f \cdot g)(x)}{z-x} = \frac{f(z) - f(x)}{z-x} g(z) + f(x) \frac{g(z) - g(x)}{z-x} \tag{29.23}$$

By Lemma 29.5, the existence of $g'(x)$ implies continuity and thus $g(z) \to g(x)$ as $z \to x$. Hereby (29.21) follows using the Sum and Product Rules for limits. □

Lemma 29.8 (Chain Rule) *Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be functions and let $x \in \operatorname{int}(\operatorname{Dom}(f))$ be such that f is differentiable at x and such that $f(x) \in \operatorname{int}(\operatorname{Dom}(f))$ and g is differentiable at $f(x)$. Then $g \circ f$ is differentiable at x and*

$$(g \circ f)'(x) = g'\big(f(x)\big) f'(x) \tag{29.24}$$

Proof. One way to prove this is to write

$$\frac{(g \circ f)(z) - (g \circ f)(x)}{z-x} = \frac{g(f(z)) - g(f(x))}{f(z) - f(x)} \, \frac{f(z) - f(x)}{z-x} \tag{29.25}$$

and then use the fact that $f(z) \to f(x)$ to take the limit using the Product Rule for limits. However, the problem with this argument occurs when $f(z) = f(x)$ for z arbitrarily close to x (and in particular, for all z when f is constant) which makes the right-hand side of (29.25) undefined. We will thus proceed using a different argument that by-passes this issue altogether.

Using the argument from the proof of Lemma 29.5, for $z \in \operatorname{Dom}(f)$ we have

$$f(z) - f(x) = \big[f'(x) + u_x(z)\big](z-x) \tag{29.26}$$

and, for $y \in \operatorname{Dom}(g)$ we have

$$g(y) - g\big(f(x)\big) = \big[g'(f(x)) + v_{f(x)}(y)\big]\big(y - f(x)\big) \tag{29.27}$$

where

$$\lim_{z \to x} u_x(z) = 0 \ \wedge \ \lim_{y \to f(x)} v_{f(x)}(y) = 0 \tag{29.28}$$

For all $z \in \operatorname{Dom}(g \circ f)$ we then get

$$\begin{aligned} g\big(f(z)\big) - g\big(f(x)\big) &= \big[g'(f(x)) + v_{f(x)}(y)\big]\big(f(z) - f(x)\big) \\ &= \big[g'(f(x)) + v_{f(x)}(y)\big]\big[f'(x) + u_x(z)\big](z-x) \end{aligned} \tag{29.29}$$

Using (29.28) along with the fact that, by Lemma 29.5, $f(z) \to f(x)$ as $z \to x$, the product of the two brackets on the right tends to $g'(f(x))f'(x)$ as $z \to x$. Dividing the expression by $z - x$, the claim follows by taking $z \to x$. □

Another useful "Rule" is:

Lemma 29.9 (Inverse function rule) *Let $f\colon \mathbb{R}\to\mathbb{R}$ be injective and let $x\in\operatorname{int}(\operatorname{Dom}(f))$ be such that $f(x)\in\operatorname{int}(\operatorname{Ran}(f))$. Assume that the inverse function f^{-1} of f (with $\operatorname{Dom}(f^{-1}) := \operatorname{Ran}(f)$) is continuous at the point $f(x)$. If $f'(x)\neq 0$, then f^{-1} is also differentiable at the point $f(x)$ and we have*

$$(f^{-1})'\big(f(x)\big) = \frac{1}{f'(x)} \tag{29.30}$$

We leave the proof of this lemma to homework. Note that no assumptions about regularity of f are made outside x. In particular, f can be discontinuous and non-monotone in $(x-\delta, x+\delta)$ for any $\delta>0$ — all we need is that f^{-1} is continuous at $f(x)$. The latter requirement may be replaced by other conditions; e.g., continuous differentiability of f.

29.4 Relation to local extrema.

One aspect that is discussed at length in Calculus is the relation of the derivative to the local minima/maxima of a function. We start with a formal definition of these concepts:

Definition 29.10 (Local minimum/maximum) *Let $f\colon X\to\mathbb{R}$ be a function on a metric space (X,ρ). Let $x\in\operatorname{Dom}(f)$. We say that f has* local minimum *at x if*

$$\exists\delta>0\,\forall z\in\operatorname{Dom}(f)\colon\ \rho(x,z)<\delta\ \Rightarrow\ f(x)\leqslant f(z) \tag{29.31}$$

and a local maximum *at x if*

$$\exists\delta>0\,\forall z\in\operatorname{Dom}(f)\colon\ \rho(x,z)<\delta\ \Rightarrow\ f(z)\leqslant f(x) \tag{29.32}$$

These are strict *local minima/maxima if the inequalities on the right are strict for $z\neq x$.*

We note that a point that is either a local minimum or a local maximum is generally refered to as a *local extremum*. We now have:

Theorem 29.11 (First derivative test) *Let $f\colon\mathbb{R}\to\mathbb{R}$ and $x\in\operatorname{int}(\operatorname{Dom}(f))$ be such that f is differentiable at x. If f has a local maximum or a local minimum at x, then $f'(x)=0$.*

Proof. We can assume that f has a local maximum (for otherwise we can work with $-f$ instead). Let $\delta>0$ be such that for all $z\in(x-\delta,x+\delta)$ we have $z\in\operatorname{Dom}(f)$ and $f(z)\leqslant f(x)$. Then for such z,

$$z>x\ \Rightarrow\ \frac{f(z)-f(x)}{z-x}\leqslant 0 \tag{29.33}$$

and so we get

$$f'^{,+}(x) := \lim_{z\to x^+}\frac{f(z)-f(x)}{z-x}\leqslant 0 \tag{29.34}$$

On the other hand,

$$z<x\ \Rightarrow\ \frac{f(z)-f(x)}{z-x}\geqslant 0 \tag{29.35}$$

yields

$$f'^{,-}(x) := \lim_{z\to x^-}\frac{f(z)-f(x)}{z-x}\geqslant 0 \tag{29.36}$$

Since $f'(x)$ is assumed to exist, the one-sided derivatives in (29.34) and (29.36) are both equal $f'(x)$ which then forces $f'(x)=0$. □

Note that if $\mathrm{Dom}(f)$ is an interval (or a union of intervals with disjoint closures) and x is an endpoint of the intervals, only one of (29.34–29.36) applies. This is the reason why we have to include these endpoints (and, generally, the boundary of the domain) when trying to find all local extrema of the function at hand.

Another remark to make is that a function may have a local extremum at a point where the derivative does not exist. An example of this is the function $f(x) := |x|$ which has a local (and in fact global) minimum at $x = 0$ yet f is not differentiable there. (The one-sided derivatives do exist, though, and obey the corresponding inequalities.)

As a final note we recall that further information about the behavior of the function at a local extremum can be obtained by studying higher-order derivatives (typically, the second derivative). We will say more in a later chapter.