28. DISCONTINUITIES

Having spent considerable amount of time discussing continuous functions, we will now take a closer look at functions at their points of discontinuity.

28.1 Discontinuities of first and second kind.

We start by a concept that exist in full generality of functions on metric spaces:

Definition 28.1 *Let $f\colon X \to Y$ be a function between metric spaces X and Y and $x \in \operatorname{Dom}(f)$. We say that f has a* removable discontinuity *at x if*

$$\lim_{z\to x} f(z) \text{ exists } \wedge \lim_{z\to x} f(z) \neq f(x) \tag{28.1}$$

The motivation for the name comes from Lemma 27.4 which shows that if f has a removable discontinuity at x then a simple re-definition of f leads to a continuous function.

The remaining discussion will be restricted to functions $f\colon \mathbb{R} \to \mathbb{R}$. As should be clear from above, having the one sided limits at x is the next best thing one can hope to have if the full limit does not exist and/or the function is not continuous. This motivates:

Definition 28.2 (Discontinuities of 1st and 2nd kind) *Let $f\colon \mathbb{R} \to \mathbb{R}$ and $x \in \operatorname{int}(\operatorname{Dom}(f))$. We say that f has a* discontinuity of the first kind *at x if*

$$f(x^+), f(x^-) \text{ exist } \wedge \left|\{f(x^-), f(x^+), f(x)\}\right| > 1 \tag{28.2}$$

We say that f has discontinuity of the second kind *at x if at least one of the limits $f(x^+)$ and $f(x^-)$ does NOT exist.*

To demonstrate these, we note:

Lemma 28.3 *Let $f\colon \mathbb{R} \to \mathbb{R}$ (with $\operatorname{Dom}(f) = \mathbb{R}$) be monotone. Then $f(x^+)$ and $f(x^-)$ exist at all $x \in \mathbb{R}$. In particular, f has no discontinuities of the second kind.*

Proof. Suppose that f is non-decreasing (otherwise take $-f$ instead of f) and let $x \in \mathbb{R}$. We claim that $f(x^+)$ exists and, in fact,

$$f(x^+) = \inf\{f(z)\colon z > x\} \tag{28.3}$$

Indeed, since f is non-decrasing, $f(x)$ is a lower bound on every value in the set and so the infimum exists proper in $\mathbb{R}$. Writing c for the infimum, it follows that, for each $\epsilon > 0$ there is z_0 such that

$$c \leqslant f(z_0) < c + \epsilon \tag{28.4}$$

Denoting $\delta := z_0 - x$, the monotonicity of f then ensures that $c \leqslant f(z) < c + \epsilon$ holds for all $z \in (x, x+\delta)$, i.e.,

$$f\big((x, x+\delta)\big) \subseteq (c-\epsilon, c+\epsilon) \tag{28.5}$$

As this applies for all $\epsilon > 0$, the definition (27.27) gives $f(x^+) = c$ as desired. The left limit is treated analogously (or turned into the above by considering $x \mapsto f(-x)$). □

We note that the above arguments can be bolstered to give us even the following:

Lemma 28.4 *Let $f\colon \mathbb{R}\to\mathbb{R}$ be non-decreasing with* $\operatorname{Dom}(f)=\mathbb{R}$. *Then $x\mapsto f(x^+)$ is right continuous while $x\mapsto f(x^-)$ is left continuous with both non-decreasing. Moreover,*

$$\forall x\in\mathbb{R}\colon\;\; f(x^-)\leqslant f(x)\leqslant f(x^+) \tag{28.6}$$

while

$$\forall x,y\in\mathbb{R}\colon\;\; x<y\;\Rightarrow\; f(x^+)\leqslant f(y^-) \tag{28.7}$$

We leave a proof of this lemma to a homework exercise. In order to give an example of a function with discontinuities of the second kind, consider the Dirichlet function $1_{\mathbb{Q}}$ defined in (27.12) that fails to have one-sided limits at every point of $\mathbb{R}$. A slightly more subtle example is the function h from (26.15) extended by zero to $(-\infty,0]$ that has discontinuity of the second kind at 0.

Proceeding in the discussion of monotone functions, we now observe:

Lemma 28.5 *Let $f\colon \mathbb{R}\to\mathbb{R}$ be monotone with* $\operatorname{Dom}(f)=\mathbb{R}$. *Then*

$$\bigl\{x\in\mathbb{R}\colon f(x^+)\neq f(x^-)\bigr\}\text{ is finite or countable} \tag{28.8}$$

Proof. Assume f to be non-decreasing, denote $A:=\{x\in\mathbb{R}\colon f(x^+)\neq f(x^-)\}$ and let $\{q_n\}_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}$. Noting that $f(x^-)<f(x^+)$ for all $x\in A$, define $\sigma\colon A\to\mathbb{N}$ by

$$\sigma(x):=\inf\bigl\{n\in\mathbb{N}\colon f(x^-)<q_n<f(x^+)\bigr\} \tag{28.9}$$

Then $f(x^-)<q_{\sigma(x)}<f(x^+)$ and, if $x<y$ are such that $x,y\in A$, we thus have

$$q_{\sigma(x)}<f(x^+)\leqslant f(y^-)<q_{\sigma(y)} \tag{28.10}$$

proving $\sigma(x)\neq\sigma(y)$. The map σ is thus an injection of A into $\mathbb{N}$, proving the claim. $\square$

We remark that while the monotonicity has entered crucially in the proof, it is actually not required for the result. Indeed, we have:

Lemma 28.6 *Let $f\colon \mathbb{R}\to\mathbb{R}$ with* $\operatorname{Dom}(f)=\mathbb{R}$ *be such that*

$$\forall x\in\mathbb{R}\colon f(x^+),f(x^-)\text{ exist} \tag{28.11}$$

Then

$$\bigl\{x\in\mathbb{R}\colon f(x^+)\neq f(x^-)\bigr\}\text{ is finite or countable} \tag{28.12}$$

Proof (ideas). Here is one idea: Writing A for the set under consideration, using the construction in (28.9) (for a given enumeration of $\mathbb{Q}$ into a sequence) we define functions $p,q,r\colon A\to\mathbb{Q}$ so that, for $x\in A$ with $f(x^-)<f(x^+)$ we have

$$f(x^-)<q<f(x^+)\;\;\wedge\;\; p(x)<x<r(x) \tag{28.13}$$

and that

$$\forall z\in(p(x),x)\colon\; f(z)<q\;\;\wedge\;\;\forall z\in(x,r(x))\colon\; f(z)>q \tag{28.14}$$

The role of p and r is interchanged for $x\in A$ with $f(x^+)<f(x^-)$. Using some case analysis we we then check that $x\mapsto(p(x),q(x),r(x))$ is an injection of A into $\mathbb{Q}^3$.

Another idea is to show that, for each $M,N\geqslant 1$, the set

$$\bigl\{x\in[-M,M]\colon |f(x^+)-f(x^-)|\geqslant 1/N\bigr\} \tag{28.15}$$

is finite. (The proof would assume the contrary and show, with the help of the Bolzano-Weierstrass theorem) the existence of a point where the function has a discontinuity of a second kind.) Then take the union over all natural $M, N \geqslant 1$. □

As is checked by the example of the Dirichlet function, none of this of course applies when the function is allowed to have discontinuities of the second kind.

28.2 How bad can the set of discontinuities get?.

In (27.23) we gave an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous at all irrational points and discontinuous, with a removable discontinuity, at all rational points. To spice things up, we construct a function in which the discontinuities are not removable.

Let $h\colon \mathbb{R} \to [0,1]$ be a function that is continuous at all points in $\mathbb{R} \smallsetminus \{0\}$ but with a discontinuity of the second kind at $x = 0$. (The function from (26.15) extended by zero provides an example.). Now set

$$f(x) := \sum_{n=0}^{\infty} 2^{-n} h(x - q_n) \tag{28.16}$$

where $\{q_n\}_{n\in\mathbb{N}}$ is a sequence enumerating $\mathbb{Q}$. Each summand $x \mapsto 2^{-n}h(x - q_n)$ is then continuous on $\mathbb{R} \smallsetminus \{q_n\}$ with a discontinuity of the second kind at $x = q_n$. Moreover, if $\epsilon > 0$ is smaller than 2^{-m+1}, we have

$$\begin{aligned} \left| f(x) - \sum_{n=0}^{m} 2^{-n} h(x - q_n) \right| &\leqslant \sum_{n=m+1}^{\infty} 2^{-m} h(x - q_n) \\ &\leqslant \sum_{n=m+1}^{\infty} 2^{-m} = 2^{-m+1} < \epsilon \end{aligned} \tag{28.17}$$

and so f is approximated by the finite sum up to error less than ϵ. Using "rules for limits," $x \mapsto \sum_{n=0}^{m} 2^{-n} h(x - q_n)$ is continuous at all $x \neq q_0, \dots, q_m$, and has a discontinuity of a second kind at $x = q_0, \dots, q_m$. This implies that f has a discontinuity of the second kind at all rational points and is continuous at all irrational points.

A natural next question is: How bad can the set of discontinuity points get? We answer this partially in the following statement:

Theorem 28.7 *Let X and Y be metric spaces and let $f\colon X \to Y$ be such that* $\operatorname{Dom}(f) = X$. *Then* $\{x \in X\colon f \text{ is NOT continuous at } x\}$ *is a countable union of closed sets.*

Proof. The proof uses the concept of "oscillation at a point" defined by

$$\operatorname{osc}_f(x) := \inf_{\delta > 0} \sup \Big\{ \rho_Y\big(f(z), f(y)\big) \colon y, z \in B_X(x, \delta) \Big\} \tag{28.18}$$

(The infimum can be replaced by the limit $\delta \to 0^+$ but the above makes sense even if the supremum is infinite.) We proceed by proving two claims:

Claim 1: For all $x \in X$, we have

$$f \text{ continuous at } x \;\Leftrightarrow\; \operatorname{osc}_f(x) = 0 \tag{28.19}$$

To prove this, assume first that f is continuous at x. Given $\epsilon > 0$, there exists $\delta > 0$ such that $f(B_X(x,\delta)) \subseteq B_Y(f(x),\epsilon)$. But then for all $y,z \in B_X(x,\delta)$, we have

$$\rho_Y\bigl(f(z),f(y)\bigr) \leqslant \rho_Y\bigl(f(z),f(x)\bigr) + \rho_Y\bigl(f(x),f(y)\bigr) < \epsilon + \epsilon = 2\epsilon \tag{28.20}$$

proving that $\operatorname{osc}_f(x) < 2\epsilon$. Since this is true for all $\epsilon > 0$, we have $\operatorname{osc}_f(x) = 0$.

Conversely, assume $\operatorname{osc}_f(x) = 0$. Given $\epsilon > 0$, there exists $\delta > 0$ such that the supremum in (28.18) is strictly less than ϵ. But taking $y := x$ then shows that $\rho_Y(f(z), f(x)) < \epsilon$ whenever $y \in B_X(x,\delta)$, proving that f is continuous at x.

Claim 2: For all $\epsilon > 0$, we have

$$\bigl\{x \in X\colon \operatorname{osc}_f(x) < \epsilon\bigr\} \text{ is open} \tag{28.21}$$

Let $x \in X$ be such that $\operatorname{osc}_f(x) < \epsilon$. Then there exists $\delta > 0$ such that the supremum in (28.18) is still strictly less than ϵ. For any $\tilde{x} \in B_X(x,\delta/2)$ we then have $B_X(\tilde{x},\delta/2) \subseteq B_X(x,\delta)$ and so

$$\begin{aligned} \operatorname{osc}_f(\tilde{x}) &\leqslant \sup\Bigl\{\rho_Y\bigl(f(z),f(y)\bigr)\colon y,z \in B_X(\tilde{x},\delta/2)\Bigr\} \\ &\leqslant \sup\Bigl\{\rho_Y\bigl(f(z),f(y)\bigr)\colon y,z \in B_X(x,\delta)\Bigr\} < \epsilon \end{aligned} \tag{28.22}$$

This gives $\operatorname{osc}_f(\tilde{x}) < \epsilon$ for all $\tilde{x} \in B_X(x,\delta/2)$, proving $B_X(x,\delta/2) \subseteq \{z \in X\colon \operatorname{osc}_f(z) < \epsilon\}$. The set $\{z \in X\colon \operatorname{osc}_f(z) < \epsilon\}$ is thus open as claimed.

With the two claims in hand, we now finish the proof: Using Claim 1 we have

$$\begin{aligned} \bigl\{x \in X\colon f \text{ is NOT continuous at } x\bigr\} &= \bigl\{x \in X\colon \operatorname{osc}_f(x) > 0\bigr\} \\ &= \bigcup_{n\in\mathbb{N}} \bigl\{x \in X\colon \operatorname{osc}_f(x) \geqslant 2^{-n}\bigr\} \end{aligned} \tag{28.23}$$

By Claim 2 the sets under intersection are closed and so we get the desired representation as a countable union of closed sets. □

We remark that sets that are countable unions of closed sets are called *type-$\mathcal{F}_\sigma$*, where "$\mathcal{F}$" is for the French word *fermé*, meaning "closed," and σ is for the union. The complement of an $\mathcal{F}_\sigma$-set is then a countable intersection of open sets, called *type-$\mathcal{G}_\delta$*, where "$\mathcal{G}$" is for the German word *Gebiet*, meaning "area," and δ is for the intersection. In order to demonstrate that the set of discontinuities of a function cannot be too bad, we prove a seemingly unrelated result whose relevance will become clear later:

Theorem 28.8 *The set $\mathbb{Q}$ is NOT a $\mathcal{G}_\delta$-set; i.e., the set of rationals cannot be written as a countable intersection of open sets.*

It is not surprising if the reader finds the statement "clearly" FALSE. Indeed, enumerating $\mathbb{Q}$ into a sequence $\{q_n\}_{n\in\mathbb{N}}$, we may take

$$O_k := \bigcup_{n\in\mathbb{N}} (q_n - 2^{-n-k}, q_n + 2^{-n-k}) \tag{28.24}$$

Then O_k is open and $\mathbb{Q} \subseteq O_k$ for each $k \in \mathbb{N}$. Since the length of each interval constituting O_k is at most 2^{-k+1}, it appears that, as $k \to \infty$, the intervals in O_k "shrink" to singletons containing just rationals and so the intersection of $\{O_k\}_{k\in\mathbb{N}}$ is $\mathbb{Q}$. However, this reasoning is faulty due to the fact that the intervals overlap and so their length tending to zero does not necessarily lead to their "collapse" to a point. (All that we can get by such arguments is that $\mathbb{Q} \subseteq \bigcap_{k\in\mathbb{N}} O_k$.)

Proof. Enumerate $\mathbb{Q}$ into a sequence $\{q_n\}_{n\in\mathbb{N}}$ and assume, for the sake of contradiction, that there are open sets $\{U_n\}_{n\in\mathbb{N}}$ such that $\mathbb{Q} = \bigcap_{n\in N} U_n$. In particular, each U_k contains $\mathbb{Q}$. We now proceed to identify a sequence of non-empty closed intervals $\{I_k\}_{k\in\mathbb{N}}$ such that

$$\forall k \in \mathbb{N}\colon\ I_{k+1} \subseteq I_k \subseteq U_k \ \wedge\ I_k \cap \{q_j \colon j \leqslant k\} = \varnothing \tag{28.25}$$

For this we need two sequences $\{n_k\}_{k\in\mathbb{N}}$ and $\{\delta_k\}_{k\in\mathbb{N}}$ defined so that

$$n_0 := 0 \wedge \delta_0 := \sup\bigl\{\delta \in (0,1)\colon (q_0 - 2\delta, q_0 + 2\delta) \subseteq U_0\bigr\} \tag{28.26}$$

and, for all $k \in \mathbb{N}$,

$$n_{k+1} := \inf\Bigl\{n > n_k \colon q_n \in (q_{n_k} - \delta_k, q_{n_k} + \delta_k)\Bigr\} \tag{28.27}$$

and

$$\delta_{k+1} := \sup\left\{\delta \in (0, 2^{-k})\colon \begin{array}{l} (q_{n_{k+1}} - 2\delta, q_{n_{k+1}} + 2\delta) \subseteq U_{k+1} \cap (q_{n_k} - \delta_k, q_{n_k} + \delta_k) \\ \forall j = 0, \dots, n_k \colon |q_j - q_{n_{k+1}}| \geqslant \delta \end{array}\right\} \tag{28.28}$$

The latter is well defined because $U_{k+1} \cap (q_{n_k} - \delta_k, q_{n_k} + \delta_k)$ contains at least one rational and is open. Now set

$$I_k := [q_{n_k} - \delta_k, q_{n_k} + \delta_k] \tag{28.29}$$

and check that the construction ensures (28.25).

To prove the claim, we now observe that the fact that $\mathbb{R}$ is complete and, by (28.25), the intervals $\{I_k\}_{k\in\mathbb{N}}$ are closed, nested and of diameter tending to zero ensures that they have a point in common. Since $I_k \subseteq U_k$, we thus get an x such that

$$x \in \bigcap_{n\in\mathbb{N}} I_k \subseteq \bigcap_{k\in\mathbb{N}} U_k = \mathbb{Q} \tag{28.30}$$

Yet the second part of (28.25) forces $x \neq q_n$ for all $n \in \mathbb{N}$, i.e., $x \notin \mathbb{Q}$, a contradiction. □

Remark 28.9 The above proof is, more or less, that of the celebrated *Baire category theorem* which is often invoked to infer the above as a corollary.

We now use this to get:

Corollary 28.10 *There exists no $f\colon \mathbb{R} \to \mathbb{R}$ (with* $\operatorname{Dom}(f) = \mathbb{R}$*) whose set of discontinuity points are the irrationals; i.e., such that* $\{x \in X\colon f \text{ is NOT continuous at } x\} = \mathbb{R} \smallsetminus \mathbb{Q}$.

Proof. If f were such a function then the set of its continuity points would be $\mathbb{Q}$. But that would force $\mathbb{Q}$ to be a $\mathcal{G}_\delta$-set which is ruled out in Theorem 28.8. □

We note that the fact that we based the above on the rationals is not important; all we needed is a set that is countable and dense. The reader may wonder if the necessary condition in Theorem 28.7 is somewhat close to sufficient. This is indeed the case, at least for real-valued function of the reals:

Lemma 28.11 *Let $A \subseteq \mathbb{R}$ be a $\mathcal{F}_\sigma$-set. Then there exists $f\colon \mathbb{R} \to \mathbb{R}$ with* $\operatorname{Dom}(f) = \mathbb{R}$ *such that* $A = \{x \in \mathbb{R}\colon f \text{ is NOT continuous at } x\}$.

Proof. It is instructive to first prove the result for A closed. In this case we set

$$f(x) := \begin{cases} 1 + 1_{\mathbb{Q}}(x) & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{28.31}$$

and check the following:

- if $x \notin A$, then $f(B(x,\delta)) = \{0\}$ for some $\delta > 0$ and so f is continuous at x,
- if $x \in \operatorname{int}(A)$, then $f(B(x,\delta)) = \{1,2\}$ for all $\delta > 0$ and so f is discontinuous at x,
- if $x \in \partial A$, then either $\{0,1\} \subseteq h(B(x,\delta))$ OR $\{0,2\} \subseteq f(B(x,\delta))$ holds for all $\delta > 0$ and so f is discontinuous at x

In short, A is the discontinuity set of f, as desired.

To deal with the general case, assume $A = \bigcup_{n\in\mathbb{N}} C_n$ for $C_n \subseteq \mathbb{R}$ closed. Taking finite unions we may assume $\{C_n\}_{n\in\mathbb{N}}$ to be nested; i.e., $\forall n \in \mathbb{N}\colon C_n \subseteq C_{n+1}$. Define

$$h_n(x) := \begin{cases} 1 + 1_{\mathbb{Q}}(x) & \text{if } x \in C_n \\ 0 & \text{if } x \notin C_n \end{cases} \tag{28.32}$$

and set

$$f(x) := \sum_{k=0}^{\infty} 8^{-k} h_k(x) \tag{28.33}$$

where the sum converges because $0 \leqslant h_k \leqslant 2$. We will also abbreviate

$$f_n(x) := \sum_{k=0}^{n} 8^{-k} h_k(x) \tag{28.34}$$

The key point to note is that, for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$,

$$\big|f_n(x) - f(x)\big| \leqslant 2 \sum_{k=n+1}^{\infty} 8^{-k} = \frac{2}{7} 8^{-n} \tag{28.35}$$

Using the "oscillation at x" defined in (28.18), this gives

$$\big|\operatorname{osc}_{f_n}(x) - \operatorname{osc}_f(x)\big| \leqslant 2 \sup_{x\in\mathbb{R}} \big|f_n(x) - f(x)\big| \leqslant \frac{4}{7} 8^{-n} \tag{28.36}$$

We now fix $n \in \mathbb{N}$ observe the following facts: If $x \notin C_n$, then there is $\delta > 0$ such that $f_n = 0$ on $B(x,\delta)$. It follows that $\operatorname{osc}_{f_n}(x) = 0$ and so

$$\forall x \in C_n\colon\ \operatorname{osc}_f(x) \leqslant \frac{4}{7} 8^{-n} \tag{28.37}$$

If in turn $x \in C_n \smallsetminus C_{n-1}$ (with the convention $C_{-1} := \emptyset$ for $n = 0$), then there is $\delta > 0$ such that $f_n = 8^{-n} h_n$ on $B(x,\delta)$. Then $\operatorname{osc}_{f_n}(x) = 8^{-n} \operatorname{osc}_{h_n}(x) \geqslant 8^{-n}$ and so

$$\forall x \in C_n \smallsetminus C_{n-1}\colon\ \operatorname{osc}_f(x) \geqslant 8^{-n} - \frac{4}{7} 8^{-n} = \frac{3}{7} 8^{-n} \tag{28.38}$$

Since also $\bigcup_{n\in\mathbb{N}} C_n = \bigcup_{n\in\mathbb{N}} (C_n \smallsetminus C_{n-1})$, hence we get

$$\forall x \in \bigcup_{n\in\mathbb{N}} C_n\colon\ \operatorname{osc}_f(x) > 0 \tag{28.39}$$

yet

$$\forall x \notin \bigcup_{n\in\mathbb{N}} C_n\colon\;\; \mathrm{osc}_f(x) = 0 \tag{28.40}$$

This proves the claim via the characterization in (28.19). □