27. Limit of functions

Having discussed continuous functions at length, we now want to move to analysis of functions that are NOT continuous. As before, we start with functions between general metric spaces and then specialize to real-valued functions.

27.1 Limit and relation to continuity.

We start with a concept that should be familiar from Calculus. (Some textbooks even treat it before continuity.)

Definition 27.1 (Limit of functions) *Let $f\colon X \to Y$ be a function between metric spaces (X, ρ_X) and (Y, ρ_Y). Let $x \in \overline{\text{Dom}(f)}$ be non-isolated and let $y \in Y$. We say that f* has limit y at x *if*

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall z \in \text{Dom}(f)\colon\ 0 < \rho_X(x,z) < \delta \Rightarrow \rho_Y\big(f(z), y\big) < \epsilon \tag{27.1}$$

We say that limit of f at x exists *if the above holds for some $y \in Y$.*

Note that (27.1) is very similar to (24.1) in the metric-space based definition of continuity. The only (but very important!) difference is that here we require "$0 <$" in the premise $0 < \rho_X(x,z) < \delta$ of the implication in (27.1) and so

$$\text{the value of } f(x)\text{, if it is defined at all, is immaterial for the limit!} \tag{27.2}$$

Modulo this fact, the concept of a limit remains quite close to that of continuity, but before we get to that, we first note:

Lemma 27.2 *If $y, \tilde{y} \in Y$ are such that* (27.1) *holds for both y and $\tilde{y}$, then $y = \tilde{y}$.*

Proof. Given the setting of Definition 27.1, fix $\epsilon := \frac{1}{2}\rho(y,\tilde{y})$. If $\epsilon > 0$, then (27.1) ensures existence of $\delta > 0$ be such that $0 < \rho_X(x,z) < \delta$ implies $\rho_Y(f(z), y) < \epsilon$ and $\delta' > 0$ such that $0 < \rho_X(x,z) < \delta'$ implies $\rho_Y(f(z),\tilde{y}) < \epsilon$. But then the fact that $x \in \overline{\text{Dom}(f)}$ gives at least one $z \in \text{Dom}(f)$ with these properties and so

$$\rho_Y(y,\tilde{y}) \leqslant \rho_Y\big(f(z),y\big) + \rho_Y\big(f(z),\tilde{y}\big) < \epsilon + \epsilon = 2\epsilon = \rho_Y(y,\tilde{y}) \tag{27.3}$$

This is absurd and so we must have $\rho_Y(y,\tilde{y}) = 0$, i.e., $y = \tilde{y}$. □

Note that, without the requirement that x be a limit point of $\text{Dom}(f)$, any y would be a limit of f at x, which is something we want to avoid. With the limit unique, we introduce the notation

$$\lim_{z \to x} f(z) = y \quad := \quad f \text{ has limit } y \text{ at } x \tag{27.4}$$

The promised connection with continuity now comes in:

Lemma 27.3 *Let $f\colon X \to Y$ be a function between metric spaces X and Y. Then for each $x \in \text{Dom}(f)$ that is NOT isolated,*

$$f \text{ continuous at } x \ \Leftrightarrow\ \lim_{z \to x} f(z) = f(x) \tag{27.5}$$

Proof. Since $\rho_Y(f(z), f(x)) = 0 < \epsilon$ for $z = x$, the point $z = x$ may be added to (27.1) when $y = f(x)$. This shows equivalence (24.1) with (27.1) for this case. □

There is even a formulation that does not require f to be defined at x:

Lemma 27.4 *Let $f\colon X \to Y$ be a function between metric spaces X and Y. Let $x \in \overline{\mathrm{Dom}(f)}$ be non-isolated and let $y \in Y$. Define $g\colon X \to Y$ with $\mathrm{Dom}(g) := \mathrm{Dom}(f) \cup \{x\}$ by*

$$g(z) := \begin{cases} f(z), & \text{if } z \in \mathrm{Dom}(f) \smallsetminus \{x\} \\ y, & \text{if } z = x \end{cases} \tag{27.6}$$

Then

$$\lim_{z \to x} f(z) = y \;\Leftrightarrow\; g \text{ is continuous at } x. \tag{27.7}$$

Proof. Again, this is a direct consequence of the definitions (24.1) and (27.1). □

We will also have a characterization of a limit using convergence of sequences:

Lemma 27.5 (AC)(Sequential characterization) *Let $f\colon X \to Y$ be a function between metric spaces X and Y. Let $x \in \overline{\mathrm{Dom}(f)}$ be non-isolated and let $y \in Y$. Then*

$$\lim_{z \to x} f(z) = y \;\Leftrightarrow\; \forall \{z_n\}_{n\in\mathbb{N}} \in (\mathrm{Dom}(f) \smallsetminus \{x\})^{\mathbb{N}} \colon\, z_n \to x \;\Rightarrow\; f(z_n) \to y \tag{27.8}$$

Proof. Abbreviate $B'_X(x,\delta) := B_X(x,\delta) \cap \mathrm{Dom}(f)$. We start with the proof of $\Rightarrow$ in (27.8). Assume $\lim_{z\to x} f(z) = y$ and let $\epsilon > 0$. Then there is $\delta > 0$ such that

$$f\big(B'_X(x,\delta) \smallsetminus \{x\}\big) \subseteq B_Y(y,\epsilon) \tag{27.9}$$

Given any $\{z_n\}_{n\in\mathbb{N}} \in (\mathrm{Dom}(f) \smallsetminus \{x\})^{\mathbb{N}}$, the convergence $z_n \to x$ implies the existence of $n_0 \geqslant 0$ such that $n \geqslant n_0$ implies $z_n \in B_X(x,\delta) \smallsetminus \{x\}$. But then $f(z_n) \in B_Y(y,\epsilon)$ for all $n \geqslant n_0$ and, since this holds for all ϵ, we have $f(z_n) \to y$.

The converse will be proved via contrapositive. Assume that $\lim_{z\to x} f(z) = y$ is FALSE. Then there is $\epsilon \geqslant 0$ such that

$$\forall \delta > 0 \colon\, B_Y(y,\epsilon) \smallsetminus f\big(B'_X(x,\delta) \smallsetminus \{x\}\big) \neq \emptyset \tag{27.10}$$

Applying this for $\delta \in \{2^{-n} \colon n \in \mathbb{N}\}$, the Axiom of Choice implies existence of $\{z_n\}_{n\in\mathbb{N}}$ such that

$$\forall n \in \mathbb{N} \colon\, z_n \in B'_X(x, 2^{-n}) \smallsetminus \{x\} \;\wedge\; f(z_n) \notin B_Y(y,\epsilon) \tag{27.11}$$

But then $z_n \to x \,\wedge\, f(z_n) \nrightarrow y$ which is the logical opposite of $z_n \to x \Rightarrow f(z_n) \to y$. This proves $\Leftrightarrow$ and thus the whole claim. □

We remark that the previous lemma is mainly used to *disprove* existence of the limit. Indeed, for that it suffices to come up with two sequences $\{z_n\}_{n\in\mathbb{N}}$ and $\{z'_n\}_{n\in\mathbb{N}}$ such that $z_n \to x$ and $z'_n \to x$ and such that $f(z_n) \to y$ and $f(z_n) \to y'$ with $y \neq y'$. (Alternatively, it suffices to show that $\rho_Y(f(z_n), f(z'_n))$ stays uniformly positive.)

As an example, consider the *Dirichlet function* $1_{\mathbb{Q}} \colon \mathbb{R} \to \mathbb{R}$ defined by

$$1_{\mathbb{Q}}(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \tag{27.12}$$

If $\{z_n\}_{n\in\mathbb{N}}$ is a sequence of rationals converging to x, then $1_{\mathbb{Q}}(z_n) \to 1$, while if $\{z'_n\}_{n\in\mathbb{N}}$ is a sequence of irrationals converging to x, then $1_{\mathbb{Q}}(z'_n) \to 0$. As both rationals and irrationals are dense, we conclude that $1_{\mathbb{Q}}$ fails to have a limit, and (by Lemma 27.3) is thus NOT continuous, at every $x \in \mathbb{R}$.

The connection with continuity (and/or sequential characterization) allows us to prove the various "rules" for limits. We give these all in one in:

Lemma 27.6 (Sum, Product and Quotient Rule for limits) *Let $f,g\colon X \to \mathbb{R}$ be functions such that x is isolated neither in $\operatorname{Dom}(f)$ nor in $\operatorname{Dom}(g)$ and such that both f and g have limits at x. Then so do the functions $f+g$ and $f\cdot g$ and we have*

$$\lim_{z\to x}(f+g)(z) = \lim_{z\to x} f(z) + \lim_{z\to x} g(z) \tag{27.13}$$

and

$$\lim_{z\to x}(f\cdot g)(z) = \lim_{z\to x} f(z) \cdot \lim_{z\to x} g(z) \tag{27.14}$$

Moreover, if $\lim_{z\to x} g(z) \neq 0$ then also f/g has a limit at x and

$$\lim_{z\to x}(f/g)(z) = \frac{\lim_{z\to x} f(z)}{\lim_{z\to x} g(z)} \tag{27.15}$$

where the limit on the left is by definition from $\operatorname{Dom}(f/g) := \{z \in \operatorname{Dom}(g)\colon g(z) \neq 0\}$.

Proof. By Lemma 27.4 and the fact that the value of a function at x is immaterial for the limit at x, we may assume that both f and g are continuous at x. The claims then follow from Lemmas 24.2–24.4 (with limits replaced by the value of the functions at x). If a reliance on the Axiom of Choice is not of concern, one can alternatively proceed via Lemma 27.5 and the corresponding "rules" for limits of sequences. □

27.2 Limsup/liminf and limit from a set.

As for the numerical sequences, once we treat functions taking values in the reals, there is an alternative description of convergence using the ordering of $\mathbb{R}$ by $\leqslant$. The starting concepts in this are:

Definition 27.7 (Limes superior and inferior) *Given $f\colon X \to \mathbb{R}$ on a metric space (X,ρ_X) and a non-isolated point $x \in \overline{\operatorname{Dom}(f)}$, let*

$$\limsup_{z\to x} f(z) := \inf_{\delta>0} \sup_{\substack{z\in\operatorname{Dom}(f)\\ 0<\rho_X(x,z)<\delta}} f(z) \tag{27.16}$$

and

$$\liminf_{z\to x} f(z) := \sup_{\delta>0} \inf_{\substack{z\in\operatorname{Dom}(f)\\ 0<\rho_X(x,z)<\delta}} f(z) \tag{27.17}$$

where the infima/suprema are taken in the extended reals $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$.

We then have:

Lemma 27.8 *Let $f\colon X \to \mathbb{R}$ and let $x \in \overline{\operatorname{Dom}(f)}$ be non-isolated. Then*

$$\liminf_{z\to x} f(z) \leqslant \limsup_{z\to x} f(z) \tag{27.18}$$

Moreover, for any $y \in \mathbb{R}$,

$$\lim_{z\to x} f(z) = y \;\Leftrightarrow\; \liminf_{z\to x} f(z) = \limsup_{z\to x} f(z) = y \tag{27.19}$$

Proof. Let $\delta, \delta' > 0$ and let $\delta'' := \min\{\delta, \delta'\}$. Then

$$\inf_{\substack{z\in \mathrm{Dom}(f)\\ 0<\rho_X(x,z)<\delta}} f(z) \leqslant \inf_{\substack{z\in \mathrm{Dom}(f)\\ 0<\rho_X(x,z)<\delta''}} f(z) \leqslant \sup_{\substack{z\in \mathrm{Dom}(f)\\ 0<\rho_X(x,z)<\delta''}} f(z) \leqslant \sup_{\substack{z\in \mathrm{Dom}(f)\\ 0<\rho_X(x,z)<\delta'}} f(z) \tag{27.20}$$

where the middle inequality uses $\{z \in \mathrm{Dom}(f) \colon 0 < \rho_X(z,x) < \delta''\} \neq \emptyset$ implied by the fact that x is non-isolated. It follows that the supremum on the right is an upper bound on $\liminf_{z\to x} f(z)$ which is then a lower bound on all of the suprema, and thus also on $\limsup_{z\to x} f(z)$, proving (27.18).

In order to prove (27.19) it suffices to observe that both sides of the equivalence are equivalent to

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall z \in B_X(x,\delta) \smallsetminus \{x\}\colon\ y - \epsilon < f(z) < y + \epsilon \tag{27.21}$$

We leave checking that fact to the reader. □

Returning to our example from (27.12), we now readily check that, for each $x \in \mathbb{R}$,

$$\liminf_{z\to x} 1_{\mathbb{Q}}(z) = 0 \ \wedge\ \limsup_{z\to x} 1_{\mathbb{Q}}(z) = 1 \tag{27.22}$$

which again shows that the limit of $1_{\mathbb{Q}}$ does not exist at any point. Another example is the function

$$f(x) := \begin{cases} \frac{1}{n+1}, & \text{if } x = q_n \text{ for some } n \in \mathbb{N},\\ 0, & \text{if } x \notin \mathbb{Q}, \end{cases} \tag{27.23}$$

where $\{q_n\}_{n\in\mathbb{N}}$ is a sequence enumerating $\mathbb{Q}$. Here

$$\liminf_{z\to x} f(z) = 0 \ \wedge\ \limsup_{z\to x} f(z) = 0. \tag{27.24}$$

Here the second part is that which is non-trivial and requires a careful argument. Details of this are left to a homework assignment.

Another variation of the concept of the limit comes from further restriction of the values that can be considered under the limit sign. This comes in:

Definition 27.9 (Limit from a set) *Let $f\colon X \to Y$ and let $A \subseteq X$. Assume that $x \in \overline{A \cap \mathrm{Dom}(f)}$ is non-isolated. We then set*

$$\lim_{\substack{z\to x\\ z\in A}} f(z) := \lim_{z\to x} f_A(z) \tag{27.25}$$

where f_A is the restriction of f to $\mathrm{Dom}(f_A) := A \cap \mathrm{Dom}(f)$.

Applications of this concept typically concern a limits of a function at points on the boundary of that set. Another example comes in:

Definition 27.10 (Left and right limits) *Let $f\colon \mathbb{R} \to Y$ and let $x \in \overline{\mathrm{Dom}(f)}$. If x is not isolated in $\mathrm{Dom}(f) \cap (x,\infty)$, then we define the* right-limit of f at x *by*

$$\lim_{z\to x^+} f(z) := \lim_{\substack{z\to x\\ z\in(x,\infty)}} f(z) \tag{27.26}$$

If in turn x is not isolated in $\mathrm{Dom}(f) \cap (-\infty,x)$, then we define the left-limit of f at x *by*

$$\lim_{z\to x^-} f(z) := \lim_{\substack{z\to x\\ z\in(-\infty,z)}} f(z) \tag{27.27}$$

Alternative notations $f(x^+)$, resp., $f(x^-)$ are used for the objects in (27.26)*, resp.,* (27.27).

We then have:

Lemma 27.11 *Let $f\colon \mathbb{R} \to Y$ and let $x \in \operatorname{int}(\operatorname{Dom}(f))$. Then for any $y \in Y$,*

$$\lim_{z\to x} f(z) = y \quad\Leftrightarrow\quad f(x^+), f(x^-) \text{ exist } \wedge\ f(x^+) = f(x^-) = y \tag{27.28}$$

(The restriction to $x \in \operatorname{int}(\operatorname{Dom}(f))$ can be weakened to x not being isolated from either side in $\operatorname{Dom}(f)$.)

We leave the simple proof of this lemma to homework. As an example, consider the function sgn: $\mathbb{R} \to \mathbb{R}$ defined by

$$\operatorname{sgn}(x) := \begin{cases} 1, & \text{if } x > 0,\\ 0, & \text{if } x = 0,\\ -1, & \text{if } x < 0. \end{cases} \tag{27.29}$$

Then the right-limit of sgn at zero equals 1, the left-limit equals -1 and, since these are not equal, the limit at zero does not exist.

The one-sided limits defined in (27.26–27.27) can be linked to corresponding concepts of continuity from right and left, respectively. Also can be defined by introducing a concept of "continuity from a set;" however, for simplicity we define these directly using these limits:

Definition 27.12 (Left and right continuity) *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and $x \in \operatorname{Dom}(f)$. Then f said to be:*

(1) left continuous *at x if x is not isolated in $\operatorname{Dom}(f) \cap (-\infty, x]$ and $f(x^-) = f(x)$*
(2) right-continuous *at x if x is not isolated in $\operatorname{Dom}(f) \cap [x, \infty)$ and $f(x^+) = f(x)$.*

Clearly, these definitions ensure:

Lemma 27.13 *A function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at $x \in \operatorname{int}(\operatorname{Dom}(f))$ if and only if the one-sided limits $f(x^+)$ and $f(x^-)$ exist and $f(x^+) = f(x^-) = f(x)$. In particular, f is continuous at such an x if and only if it is both left and right continuous at x.*

We leave the elementary proof of this lemma to the reader.