

## 25. UNIFORM AND CAUCHY CONTINUITY

We have seen above that continuous functions are exactly those that convert convergent sequences to convergent sequences. A natural variation on this is: What functions convert Cauchy sequences to Cauchy sequences. Another question we are interested in concerns the dependence of  $\delta$  in the definition of continuity on the underlying point and  $\epsilon$ . Both of these lead to refinements of continuity we will discuss next.

**25.1 Refined versions of continuity.**

We start with a refinement of the sequential characterization of continuity:

**Definition 25.1** (Cauchy continuity) *Let  $f: X \rightarrow Y$  be a function between metric spaces  $X$  and  $Y$ . We say that  $f$  is Cauchy continuous if*

$$\forall \{x_n\}_{n \in \mathbb{N}} \in \text{Dom}(f)^{\mathbb{N}}: \{x_n\}_{n \in \mathbb{N}} \text{ Cauchy (in } X) \Rightarrow \{f(x_n)\}_{n \in \mathbb{N}} \text{ Cauchy (in } Y) \quad (25.1)$$

*i.e., if  $f$  turns Cauchy sequences to Cauchy sequences.*

We immediately note:

**Lemma 25.2** (AC) *A Cauchy continuous function is continuous.*

*Proof.* Let  $f: X \rightarrow Y$  be Cauchy continuous and let  $x \in \text{Dom}(f)$ . Consider a sequence  $\{x_n\}_{n \in \mathbb{N}} \in \text{Dom}(f)^{\mathbb{N}}$  such that  $x_n \rightarrow x$  and define  $\{y_n\}_{n \in \mathbb{N}}$  by  $y_{2n} := x_n$  and  $y_{2n+1} := x$ . Then also  $y_n \rightarrow x$ . But then  $\{f(y_n)\}_{n \in \mathbb{N}}$  is Cauchy which means

$$\rho_Y(f(x_n), f(x)) = \rho_Y(f(y_{2n}), f(y_{2n+1})) \rightarrow 0 \quad (25.2)$$

Hence  $f(x_n) \rightarrow f(x)$ . By Theorem 24.7 (which is where we need to call upon the Axiom of Choice),  $f$  is continuous at  $x$ .  $\square$

However, the converse does not hold. Indeed consider a function  $f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  defined by

$$f(x) := \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x < 0, \end{cases} \quad (25.3)$$

Then  $f$  is continuous yet not Cauchy continuous because the sequence  $x_n := (-2)^{-n}$  is Cauchy yet  $\{f(x_n)\}_{n \in \mathbb{N}}$  is NOT.

The concept of Cauchy continuity is somewhat special and in practice we typically use different (and slightly stronger) concepts that address directly continuity in a more quantitative way. The simplest of these is:

**Definition 25.3** (Uniform continuity) *A function  $f: X \rightarrow Y$  between metric spaces  $(X, \rho_X)$  and  $(Y, \rho_Y)$  is said to be uniformly continuous if*

$$\forall \epsilon > 0 \exists \delta > 0 \forall x, y \in X: \rho_X(x, y) < \delta \Rightarrow \rho_Y(f(x), f(y)) < \epsilon \quad (25.4)$$

As it turns out, this differs from the definition of continuity in a rather inconspicuous yet very important way. Indeed, a function  $f: X \rightarrow Y$  is continuous if

$$\forall \epsilon > 0 \forall x \in X \exists \delta > 0 \forall y \in X: \rho_X(x, y) < \delta \Rightarrow \rho_Y(f(x), f(y)) < \epsilon \quad (25.5)$$

which differs from (25.4) by a swap of  $\exists \delta > 0$  and  $\forall x \in X$  quantifiers. This swap amounts to the fact that  $\delta$  in (25.5) may depend on  $x$  yet in (25.4) same  $\delta$  must work for all  $x$  (and  $y$ ) simultaneously. This immediately gives:

**Lemma 25.4** *A uniformly continuous function is continuous.*

Again, the converse to this fails as witnessed by the example

$$f(x) := x^2 \tag{25.6}$$

with  $\text{Dom}(f) = \mathbb{R}$ . Indeed,  $(x + \delta)^2 - x^2 = 2x\delta + \delta^2$  which cannot be made smaller than  $\epsilon$  without restricting the size of  $x$ . We in fact have a stronger conclusion:

**Lemma 25.5** *A uniformly continuous function is Cauchy continuous.*

*Proof.* Let  $f: X \rightarrow Y$  be uniformly continuous. Given  $\epsilon > 0$  let  $\delta > 0$  be such that  $\rho_X(x, y) < \delta$  implies  $\rho_Y(f(x), f(y)) < \epsilon$ . Given a Cauchy sequence  $\{x_n\}_{n \in \mathbb{N}}$ , there exists  $n_0 \geq 0$  such that  $\rho_X(x_n, x_m) < \delta$  for all  $n, m \geq n_0$ . But then  $\rho_Y(f(x_n), f(x_m)) < \epsilon$  for all  $n, m \geq n_0$ . As this holds for all  $\epsilon > 0$ ,  $\{f(x_n)\}_{n \in \mathbb{N}}$  is Cauchy as desired.  $\square$

The example (25.6) shows that the converse to this does not hold; indeed, the function there is Cauchy continuous yet not uniformly continuous. However, this would not be the case if we restrict that  $f$  to any bounded subset of  $\mathbb{R}$ . Indeed, we have:

**Lemma 25.6** *Let  $X$  and  $Y$  be metric spaces with  $X$  compact. Then for all  $f: X \rightarrow Y$ :*

$$f \text{ continuous} \Rightarrow f \text{ uniformly continuous} \tag{25.7}$$

(Note that, thanks to Lemma 25.4, we can even write  $\Leftrightarrow$  in (25.7).)

*Proof #1 (based on sequences).* Suppose  $X$  is sequentially compact and assume, for an argument by contrapositive, that  $f: X \rightarrow Y$  is NOT uniformly continuous. This means that there exists  $\epsilon > 0$  and sequences  $\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}}$  such that

$$\forall n \in \mathbb{N}: \rho_X(x_n, y_n) < 2^{-n} \wedge \rho_Y(f(x_n), f(y_n)) \geq \epsilon \tag{25.8}$$

Sequential compactness gives us a subsequence  $\{n_k\}_{k \in \mathbb{N}}$  and  $x \in X$  such that  $n_k \rightarrow \infty$  and  $x_{n_k} \rightarrow x$ . But then also  $y_{n_k} \rightarrow x$ . If  $f$  were continuous at  $x$ , then this would imply  $f(x_{n_k}) \rightarrow f(x)$  and  $f(y_{n_k}) \rightarrow f(x)$ , but that contradicts the inequality  $\rho_Y(f(x_{n_k}), f(y_{n_k})) \geq \epsilon$ . So  $f$  is NOT continuous, proving the claim by contrapositive.  $\square$

*Proof #2 (based on open covers).* Assume  $X$  to be compact and let  $f: X \rightarrow Y$  be continuous. Given  $\epsilon > 0$ , for each  $x \in X$  let

$$\Delta_x := \{\delta \in (0, 1): f(B_X(x, 2\delta)) \subseteq B_Y(f(x), \epsilon/2)\} \tag{25.9}$$

The continuity of  $f$  ensures that  $\Delta_x \neq \emptyset$  for all  $x \in X$ . This, along with the fact that  $x \in B_X(x, \delta)$  once  $\delta > 0$  shows that  $\{B_X(x, \delta): x \in X \wedge \delta \in \Delta_x\}$  is an open cover of  $X$ . The assumed compactness of  $X$  then yields existence of a finite subcover, which means that there exists  $n \in \mathbb{N}$ , points  $z_0, \dots, z_n \in X$  and reals  $\delta_0, \dots, \delta_n \in (0, 1)$  such that

$$X = \bigcup_{i=0}^n B_X(z_i, \delta_i) \wedge \forall i = 0, \dots, n: \delta_i \in \Delta_{z_i} \tag{25.10}$$

Define  $\delta := \min_{i=0, \dots, n} \delta_i$  and note that  $\delta > 0$ . Let  $x, y \in X$  be two points with  $\rho_X(x, y) < \delta$ . Denoting

$$i := \min\{j = 0, \dots, n : x \in B(z_j, \delta_j)\} \quad (25.11)$$

we get

$$\rho(y, z_i) \leq \rho(x, y) + \rho(x, z_i) < \delta + \delta_i \leq 2\delta_i \quad (25.12)$$

and, using also  $\rho_X(x, z_i) < \delta \leq \delta_i \leq 2\delta_i$ , we conclude  $x, y \in B_X(z_i, 2\delta_i)$ . But  $\delta_i \in \Delta_{z_i}$  implies  $f(x), f(y) \in B_Y(f(z_i), \epsilon/2)$  and so

$$\rho_Y(f(x), f(y)) \leq \rho_Y(f(x), f(z_i)) + \rho_Y(f(z_i), f(y)) < \epsilon/2 + \epsilon/2 = \epsilon \quad (25.13)$$

This proves uniform continuity of  $f$  on  $X$ .  $\square$

The assumption of compactness of  $X$  can be relaxed and the conclusion strengthened to the following form:

**Lemma 25.7** *Let  $X$  and  $Y$  be metric spaces with  $X$  totally bounded. Then for all  $f: X \rightarrow Y$ :*

$$f \text{ Cauchy continuous} \Rightarrow f \text{ uniformly continuous} \quad (25.14)$$

We leave the proof of this lemma to homework.

## 25.2 Continuous extension and consequences.

A very important reason for dealing with all of the above concepts comes in:

**Theorem 25.8 (AC)(Existence/uniqueness of continuous extension)** *Let  $X$  and  $Y$  be metric spaces,  $A \subseteq X$  a non-empty set and  $f: A \rightarrow Y$  a function with  $\text{Dom}(f) = A$ . Assume:*

- (1)  $A$  is dense in  $X$ ; i.e.,  $\bar{A} = X$ ,
- (2)  $Y$  is complete,
- (3)  $f$  is uniformly continuous (or even just Cauchy continuous).

Then there exists a continuous function  $\bar{f}: X \rightarrow Y$  such that

$$\forall x \in A: \bar{f}(x) = f(x) \quad (25.15)$$

Moreover,  $\bar{f}$  is unique in the sense that if  $g: X \rightarrow Y$  is a continuous function with  $\text{Dom}(g) = X$  and  $g = f$  on  $A$ , then  $g = \bar{f}$  on  $X$ .

*Proof.* Let  $(X, \rho_X)$  and  $(Y, \rho_Y)$  be metric spaces as is the setting of the theorem and assume that  $A$  and  $Y$  are such that the conditions (1-2) above apply. Let  $f: A \rightarrow Y$  be a function with  $\text{Dom}(f) = A$  and assume that  $f$  is Cauchy continuous (which is implied by uniform continuity). We start with the construction of  $\bar{f}$ .

Recall that  $f: X \rightarrow Y$  with  $\text{Dom}(f) = A$  is technically a relation  $G \subseteq X \times Y$  of the specific form

$$G := \{(x, f(x)) \in X \times Y : x \in A\} \quad (25.16)$$

that we typically refer to as the graph of  $f$ . Next note that  $X \times Y$  is a metric space relative to the metric

$$\rho((x, y), (\tilde{x}, \tilde{y})) = \rho_X(x, \tilde{x}) + \rho_Y(y, \tilde{y}) \quad (25.17)$$

This permits us to consider the closure  $\bar{G}$  of  $G$  in  $X \times Y$ . We now claim:

*Claim 1:  $\bar{G}$  is the graph of a function*

Indeed, suppose  $(x, y), (x, \tilde{y}) \in \overline{G}$ . Then the fact that these are adherent points of  $G$  imply (by the AC) existence of sequences  $\{x_n\}_{n \in \mathbb{N}}, \{\tilde{x}_n\}_{n \in \mathbb{N}} \in A^{\mathbb{N}}$  such that

$$(x_n, f(x_n)) \rightarrow (x, y) \wedge (\tilde{x}_n, f(\tilde{x}_n)) \rightarrow (x, \tilde{y}) \quad (25.18)$$

with the convergences in the metric space  $(X \times Y, \rho)$ . But this means that  $x_n \rightarrow x$  and  $\tilde{x}_n \rightarrow x$  and, for the sequence  $\{z_n\}_{n \in \mathbb{N}}$  such that

$$\forall n \in \mathbb{N}: z_{2n} = x_n \wedge z_{2n+1} = \tilde{x}_n \quad (25.19)$$

also  $z_n \rightarrow x$ . But the fact that  $f$  is Cauchy then implies that  $\{f(z_n)\}_{n \in \mathbb{N}}$  is Cauchy and, since also  $f(x_n) \rightarrow y$  and  $f(\tilde{x}_n) \rightarrow \tilde{y}$ , shows

$$\rho_Y(y, \tilde{y}) = \lim_{n \rightarrow \infty} \rho_Y(f(x_n), f(\tilde{x}_n)) = \lim_{n \rightarrow \infty} \rho_Y(f(z_{2n}), f(z_{2n+1})) = 0 \quad (25.20)$$

This proves that

$$\forall (x, y), (\tilde{x}, \tilde{y}) \in \overline{G}: \tilde{x} = x \Rightarrow y = \tilde{y} \quad (25.21)$$

and so  $\overline{G}$  is the graph of a function. Let us denote this function by  $\bar{f}$ . Since  $G \subseteq \overline{G}$ , we get  $\bar{f} = f$  on  $A$ , proving (25.15). Next we note:

*Claim 2:  $\text{Dom}(\bar{f}) = X$*

To prove this, let  $x \in X$ . Then by (1) above  $x$  is adherent to  $A$  and so (by the AC) there exists  $\{x_n\}_{n \in \mathbb{N}} \in A^{\mathbb{N}}$  such that  $x_n \rightarrow x$ . The Cauchy continuity of  $f$  implies that  $\{f(x_n)\}_{n \in \mathbb{N}}$  is a Cauchy sequence and so, by the completeness of  $Y$  assumed in (2) above, there exists  $y \in Y$  such that  $f(x_n) \rightarrow y$ . But then  $(x_n, f(x_n)) \rightarrow (x, y)$  in  $X \times Y$  proving that  $(x, y) \in \overline{G}$  and thus  $x \in \text{Dom}(\bar{f})$ .

*Claim 3:  $\bar{f}$  is continuous*

The function  $\bar{f}$  is an extension of  $f$  to all points of  $X$ . To prove continuity let  $x \in X$  and assume  $\{x_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}}$  is such that  $x_n \rightarrow x$ . By the construction of  $\bar{f}$ , there exists  $\{\tilde{x}_n\}_{n \in \mathbb{N}} \in A^{\mathbb{N}}$  such that

$$\forall n \in \mathbb{N}: \rho_X(\tilde{x}_n, x_n) < 2^{-n} \wedge \rho_Y(f(\tilde{x}_n), \bar{f}(x_n)) < 2^{-n} \quad (25.22)$$

But  $\tilde{x}_n \rightarrow x$  implies  $\{f(\tilde{x}_n)\}_{n \in \mathbb{N}}$  is Cauchy and, by the completeness of  $Y$ , convergent. Since the limit of  $\{(\tilde{x}_n, f(\tilde{x}_n))\}_{n \in \mathbb{N}}$  must lie in  $\overline{G}$ , hence we get

$$(\tilde{x}_n, f(\tilde{x}_n)) \rightarrow (x, \bar{f}(x)) \quad (25.23)$$

and so  $f(\tilde{x}_n) \rightarrow \bar{f}(x)$ . But then (25.22) gives  $\bar{f}(x_n) \rightarrow \bar{f}(x)$ , proving continuity of  $\bar{f}$  at  $x$ .

For the uniqueness it suffices to note that two continuous functions defined on a closure of the set  $A$  agree once they agree on  $A$ . We leave this detail to the reader.  $\square$

The requirement that  $f$  is uniformly (or at least Cauchy) continuous is absolutely crucial. Indeed, the function  $f: \mathbb{Q} \rightarrow \mathbb{R}$  defined in (24.2) is continuous yet it admits no continuous extension to  $\overline{\mathbb{Q}} = \mathbb{R}$ . There is, in fact, a result called the *Blumberg theorem* which says that every function  $f: \mathbb{R} \rightarrow \mathbb{R}$ , regardless of how bad it is, can be restricted to a dense set  $D \subseteq \mathbb{R}$  such that the restriction is continuous (on  $D$ ).

While working with uniform continuity instead of Cauchy continuity allows us to avoid the use of the Axiom of Choice in the proof of Claim 1, it does not seem to do that for the rest of the claim, due to the fact that the completeness of  $Y$  requires working with Cauchy sequences to begin with. The continuity of the extension is easier to

prove for uniformly continuous  $f$ ; indeed, if  $\delta > 0$  is such that  $\rho_X(x, y) < \delta$  implies  $\rho_Y(f(x), f(y)) < \epsilon$  for all  $x, y \in A$ , then passing this through the limits yields the same albeit with “ $< \epsilon$ ” replaced by “ $\leq \epsilon$ ” — which implies continuity and even gives:

**Corollary 25.9** *Assuming that  $f$  in Theorem 25.8 is uniformly continuous (and all other conditions hold), then so is the extension  $\tilde{f}$ .*

To demonstrate the power of the extension theorem, let us prove one more time the existence of the exponential function:

**Lemma 25.10** *Let  $a > 0$  be real and let  $f: \mathbb{Q} \rightarrow \mathbb{R}$  be defined for  $x = p/q \in \mathbb{Q}$  with  $p, q \in \mathbb{Z}$  and  $q \neq 0$  by  $f(x) := \sqrt[q]{a^p}$ . Then  $f$  is Cauchy continuous and thus extends continuously to a unique continuous function on  $\mathbb{R}$  (written as  $x \mapsto a^x$ ).*

*Proof.* Assume without (much) loss of generality that  $a \geq 1$ . We will rely on the fact that  $a^{x+y} = a^x a^y$  which is checked algebraically for  $x, y \in \mathbb{Q}$ . This shows

$$a^y - a^x = a^x(a^{y-x} - 1) \quad (25.24)$$

Next note that for each  $\delta \in (0, 1)$  there is  $N(\delta) \in \mathbb{N}$  such that  $1 - \delta < a^{\frac{1}{N(\delta)+1}} < 1 + \delta$ . Indeed, if NOT then either  $a > (1 + \delta)^{1+n}$  for all  $n \geq 0$  or  $a < (1 - \delta)^{n+1}$  for all  $n \geq 0$  which is impossible due to the fact that  $a$  is positive and finite (prove this!).

Now let  $\{x_n\}_{n \in \mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$  be a Cauchy sequence. This sequence is bounded by, say,  $M \in \mathbb{N}$ . Pick  $\epsilon > 0$  and set  $\delta := \epsilon a^{-M}$ . The fact that  $\{x_n\}_{n \in \mathbb{N}}$  is Cauchy implies

$$\exists n_0 \in \mathbb{N} \forall n, m \geq n_0: |x_n - x_m| < \frac{1}{N(\delta) + 1} \quad (25.25)$$

But then for  $m, n \geq n_0$  the above shows

$$|f(x_m) - f(x_n)| = a^{x_m} |a^{x_n - x_m} - 1| < a^M \delta = \epsilon \quad (25.26)$$

thus proving that  $\{f(x_n)\}_{n \in \mathbb{N}}$  is Cauchy. Theorem 25.8 now yields the desired unique continuous extension of  $f$  to all of  $\mathbb{R}$ .  $\square$

### 25.3 Quantifying uniformity.

To finish the discussion of uniform continuity, we introduce a couple of standard notions that give a quantitative form of the dependence of  $\delta$  on  $\epsilon$  in (25.4).

**Definition 25.11** (Lipschitz/Hölder functions) *Let  $f: X \rightarrow Y$  be a function between metric spaces  $(X, \rho_X)$  and  $(Y, \rho_Y)$ . We say that  $f$  is Lipschitz continuous if*

$$\exists \lambda > 0 \forall x, y \in X: \rho_Y(f(x), f(y)) \leq \lambda \rho_X(x, y) \quad (25.27)$$

*and, given  $\alpha > 0$ , is  $\alpha$ -Hölder continuous if*

$$\exists \lambda > 0 \forall x, y \in X: \rho_Y(f(x), f(y)) \leq \lambda \rho_X(x, y)^\alpha \quad (25.28)$$

Note that Lipschitz continuity is a special case of Hölder continuity (corresponding to  $\alpha := 1$ ) but the vernacular is used in this form throughout mathematics. Both concepts give a quantitative form of dependence of  $\delta$  on  $\epsilon$  in (25.4); namely:

$$\epsilon = \lambda \delta \text{ for (25.27) and } \epsilon = \lambda \delta^\alpha \text{ in (25.28)} \quad (25.29)$$

A Lipschitz/Hölder continuous function is uniformly continuous and thus continuous (and also Cauchy continuous).

The smallest constant  $\lambda$  that one can put into the expressions (25.27–25.28) is sometimes called the Lipschitz/Hölder norm. The cases with  $\alpha \in (0, 1]$  are most natural because  $x, y \mapsto \rho_X(x, y)^\alpha$  is a metric and because this is pretty much all there is for functions  $\mathbb{R} \rightarrow \mathbb{R}$ ; indeed, an  $\alpha$ -Hölder function  $\mathbb{R} \rightarrow \mathbb{R}$  (relative to Euclidean metric) for  $\alpha > 1$  is necessarily constant. (We leave a proof this to homework.)

Both properties above try to estimate the distance of function values by a function of the distance of the arguments. This naturally leads to the following generalization:

**Definition 25.12** (Modulus of continuity) *Given a function  $f: X \rightarrow Y$  between metric spaces  $(X, \rho_X)$  and  $(Y, \rho_Y)$ , a continuous non-decreasing map  $\omega: [0, \infty) \rightarrow [0, \infty)$  such that  $\omega(0) = 0$  and  $\omega(t) > 0$  for  $t > 0$  is a modulus of continuity of  $f$  if*

$$\forall x, y \in X: \rho_Y(f(x), f(y)) \leq \omega(\rho_X(x, y)) \quad (25.30)$$

So  $\omega(t) = \lambda t$  in (25.27) and  $\omega(t) = \lambda t^\alpha$  in (25.28). The objective above is of course to find a “best” function  $\omega$  that works for the given  $f$ ; if for each  $\lambda \in (0, 1)$  there is a pair  $x, y \in X$  with  $\rho_X(x, y) > 0$  and  $\rho_Y(f(x), f(y)) \geq \lambda \omega(\rho_X(x, y))$ , then we sometimes talk about *the* modulus of continuity of  $f$ . (Other interpretations of this term do exist, though.) Another way to study the behavior of functions is using:

**Definition 25.13** (Oscillation) *Let  $f: X \rightarrow Y$  be a function between metric spaces  $(X, \rho_X)$  and  $(Y, \rho_Y)$  and let  $A \subseteq X$ . The oscillation of  $f$  on  $A$  is the function  $r \mapsto \text{osc}_f(A, r)$  where*

$$\text{osc}_f(A, r) := \sup \left\{ \rho_Y(f(x), f(y)) : x, y \in A \wedge \rho_X(x, y) < r \right\} \quad (25.31)$$

Here  $r$  is restricted to positive reals.

We note that the Lipschitz/Hölder norm, the oscillation and the modulus of continuity are just ways to express continuity properties (or lack thereof) of a function in a condensed and quantitative manner. There are no limits to creativity in designing other concepts with a similar purpose.