l'Hôspital's Rule & Taylor's theorem

Thm (l'Hôspital's rule, J. Benoulli 1694)

Let $f, g : (a-\delta, a+\delta) \to \mathbb{R}$ (for $a \in \mathbb{R}$, $\delta > 0$) be continuous on $(a-\delta, a+\delta)$ and diff. on $(a-\delta, a+\delta) \setminus \{a\}$. Assume

$$f(a) = 0 = g(a) \wedge \forall x \in (a-\delta, a+\delta) \setminus \{a\}: \ g(x) \neq 0 \wedge g'(x) \neq 0.$$

Then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ exists} \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} \text{ exists and } = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Pf Assume $\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ exists.

Pick $x > a$. Then by Cauchy's MVT

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} \overset{\exists y \in (a,x)}{=} \frac{f'(y)}{g'(y)}$$

So $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow \forall y \in (a,x) : \left| \frac{f'(y)}{g'(y)} - \lim_{x \to a} \frac{f'(x)}{g'(x)} \right| < \varepsilon$

So $\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$.

Same for limit $x \to a^-$.

Note Proof can be modified to

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$$

Ex (FAKE)

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1$$

Ex (FAKE)

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{x \to 0} \frac{\log(a)\, a^x}{1} = \log(a)$$

Ex (GOOD) $\alpha, \beta \in \mathbb{R}(\mathbb{Q})$

$$\lim_{x \to 1} \frac{x^\alpha - 1}{x^\beta - 1} = \lim_{x \to 1} \frac{\alpha x^{\alpha - 1}}{\beta x^{\beta - 1}} = \frac{\alpha}{\beta}$$

Taylor's Thm (requires

$f \mapsto f'$ can be iterated t

$f''(x) := (f')'(x)$,

Def Given $f : \mathbb{R} \to \mathbb{R}$ with d

the sequence $\{f^{(n)}\}_{n \in \mathbb{N}}$ of

is defined as:

- $f^{(0)}(x) = f(x) \quad x \in D$
- $\forall n \in \mathbb{N}$: $f^{(n+1)}(x) := (f^{(n)})'(x$
  for $x \in Dom(f^{(n+1)}) := \{$

We call $f^{(n)}$ the n-th derivative of

Leibnitz notation

$\frac{f'(x)}{g'(x)}$ exists.

. Then by Cauchy's MVT

$\frac{f(x)-f(a)}{g(x)-g(a)} \overset{\exists y\in(a,x)}{=} \frac{f'(y)}{g'(y)}$

$|x-a|<\delta \Rightarrow$

$\exists\delta>0$ s.t. $\forall y\in(a,x)$: $\left|\frac{f'(y)}{g'(y)} - \lim_{x\to a}\frac{f'(x)}{g'(x)}\right|<\varepsilon$

$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}$.

limit $x\to a^-$.

... be modified to

$\lim f(x)=0=\lim_{x\to a} g(x)$

Ex (FAKE)

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$$\lim_{x\to 0}\frac{a^x-1}{x} = \lim_{x\to 0}\frac{\log(a)\,a^x}{1} = \log(a)$$

Ex (GOOD) $\alpha,\beta\in\mathbb{R}(\mathbb{Q})$

$$\lim_{x\to 1}\frac{x^\alpha-1}{x^\beta-1} = \lim_{x\to 1}\frac{\alpha x^{\alpha-1}}{\beta x^{\beta-1}} = \frac{\alpha}{\beta}$$

Taylor's Thm (requires higher derivatives)

$f\mapsto f'$ can be iterated to define

$f''(x) := (f')'(x)$, $f'''(x) = (f'')'(x)$ etc.

Def Given $f:\mathbb{R}\to\mathbb{R}$ with domain $\mathrm{Dom}(f)$, the sequence $\{f^{(n)}\}_{n\in\mathbb{N}}$ of functions $\mathbb{R}\to\mathbb{R}$ is defined as:

- $f^{(0)}(x) = f(x)$ $\quad x\in\mathrm{Dom}(f^{(0)}) = \mathrm{Dom}(f)$
- $\forall n\in\mathbb{N}$: $f^{(n+1)}(x) := (f^{(n)})'(x)$

for $x\in\mathrm{Dom}(f^{(n+1)}) := \{x\in \mathrm{int}\,\mathrm{Dom}(f^{(n)}) : f^{(n)} \text{ diff. at } x\}$

We call $f^{(n)}$ the n-th derivative of f.

Leibnitz notation ... $\frac{d^n f}{dx^n}$

Thm (2nd derivative test)

Let $f: \mathbb{R} \to \mathbb{R}$, $x \in \operatorname{int} \operatorname{Dom}(f)$ be s.t. $\exists \delta > 0$: f is differentiable on $(x-\delta, x+\delta)$. Assume $f''(x)$ exists.

Then $f''(x) > 0 \wedge f'(x) = 0 \Rightarrow$ f has strict local minimum at x

$(\exists \delta > 0\ \forall y \in (x-\delta, x+\delta) \setminus \{x\}: f(y) > f(x))$

Pf $f''(x) > 0$,

$\Rightarrow \exists \delta' > 0\ \forall z \in (x-\delta', x): f'(z) < f'(x)$

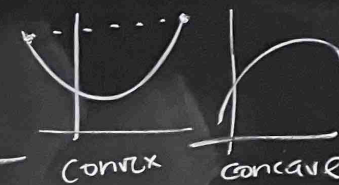
$\forall z \in (x, x+\delta'): f'(x) < f'(z)$

So $f'(x) = 0 \Rightarrow \forall z \in (x-\delta', x): f'(z) < 0$

$\forall z \in (x, x+\delta'): f'(z) > 0$

MVT $\Rightarrow \forall z \in (x-\delta', x+\delta') \setminus \{x\}: f(z) > f(x)$. $\boxtimes$

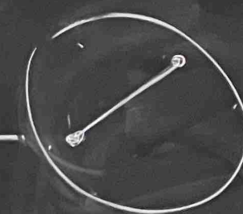
Relation to convexity



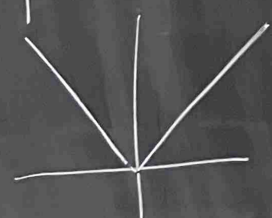
Def V = vector space. We say $f: V \to \mathbb{R}$ is convex if $\forall x, y \in \mathrm{Dom}(f)$ $\forall \alpha \in [0,1]$:

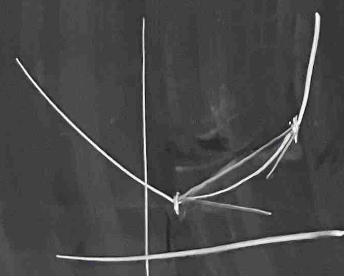
$$\alpha x + (1-\alpha) y \in \mathrm{Dom}(f) \wedge f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y)$$

We say that f is concave if $-f$ is convex



Ex $f(x) = |x|$





Lemma Let $a<b$ be reals, $f:(a,b)\to\mathbb{R}$ convex.

Then (1) $\forall x\in(a,b)$: $f'^{,-}(x)$, $f'^{,+}(x)$ exist.

(2) $\forall x\in(a,b)$: $f'^{,-}(x)\le f'^{,+}(x)$

(3) $\forall x,y\in(a,b)$:
$$x<y \Rightarrow f'^{,+}(x)\le \frac{f(y)-f(x)}{y-x}\le f'^{,-}(y)$$

(4) $\forall x\in(a,b)$:
$$\longrightarrow \quad f''(x) \text{ exists} \Rightarrow f''(x)\ge 0.$$

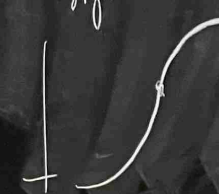
Pf: HW.

Lemma Let $f:(a,b)\to\mathbb{R}$ be twice differentiable on (a,b). The

$$\left(\forall x\in(a,b): f''(x)\ge 0\right)\Rightarrow f \text{ is convex}$$

Pf: HW

Note Points where f'' changes sign are called <u>infexion point</u>



Ex $f(x)=x^3$

Taylor's Th

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be open int

$f: I\to\mathbb{R}$

on I. The

$f(x)-$

$\ldots) \to \mathbb{R}$ convex.

$\ldots(x)$ exist.

$\ldots \le f'^{+}(x)$

$\ldots \frac{\ldots - f(x)}{\ldots - x} \le f'^{-}(y)$

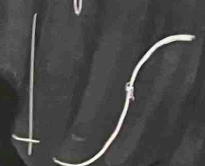
$\ldots 0.$

Lemma Let $f:(a,b)\to\mathbb{R}$ be twice differentiable on (a,b). Then

$(\forall x\in(a,b): f''(x)\ge 0) \Rightarrow f$ is convex

Pf. HW

Note Points where f'' changes sign are called **inflexion point**



Ex $f(x)=x^3$

Taylor's Thm (generalizes linear approximation)

Thm (Taylor 1715, Gregory 1691) Let $I \subseteq \mathbb{R}$ be open interval, $n\in\mathbb{N}$, $a,x\in I$. Assume $f: I\to\mathbb{R}$ $(\mathrm{Dom}(f)=I)$ is $(n+1)$-times differentiable on I. Then $\exists\, z\in(\min\{a,x\}, \max\{a,x\})$:

$$f(x) - \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{=\,P_n(x)} = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

$P_n(x)$ = **n-th order Taylor polynomial**

Pf Assume $x > a$. **Key fact:** $\forall k=0,\dots,n$: $f^{(k)}(a) = P_n^{(k)}(a)$

Denote: $A := \dfrac{f(x) - P_n(x)}{(x-a)^{n+1}}$ and

$$h(t) := f(t) - P_n(t) - A\,(t-a)^{n+1}, \quad t \in [a, x].$$

Observe: $\forall k=0,\dots,n$: $h^{(k)}(a) = 0$

claim $\forall k=0,\dots,n+1$ $\exists z_k \in (a, x]$: $h^{(k)}(z_k) = 0$

Pf $k=0$ from $h(x) = 0$ so $z_0 := x$.

Assume TRUE for $k \le n$. Then

$$h^{(k)}(a) = 0 = h^{(k)}(z_k) \xRightarrow{\text{Rolle's MVT}} \exists z_{k+1} \in (a, z_k) : h^{(k+1)}(z_{k+1}) = 0$$

Now check:

$$0 = h^{(n+1)}(z_{n+1}) = f^{(n+1)}(z_{n+1}) - \frac{f(x) - P_n(x)}{(x-a)^{n+1}}\,(n+1)!$$