Derivative & applications

$$f(z) - f(x) = [f'(x) + u_x(z)](z-x)$$

recall $f'(x) =$ der. of f at x

"rules": sum, product, quotient

chain rule $(g\circ f)'(x) = g'(f(x))\, f'(x)$, $\left(\frac{d\, g\circ f}{dx} = \frac{dg}{df}\frac{df}{dx}\right)$

Lemma (Inverse function rule) Let $f: \mathbb{R} \to \mathbb{R}$, $x \in \operatorname{Int} \operatorname{Dom}(f)$ s.t. $f(x) \in \operatorname{Int} \operatorname{Ran}(f)$. Assume f injective s.t. f^{-1} is continuous at $f(x)$. Assume also $f'(x)$ exists and obeys $f'(x) \neq 0$. Then $(f^{-1})'(f(x))$ exists and obeys

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

Ex $f(x) = x^q$ ... inverse $f^{-1}(x) = x^{1/q}$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{q(f^{-1}(x))^{q-1}} = \frac{1}{q(x^{1/q})^{q-1}} = \frac{1}{q}x^{1/q-1}$$

Ex $f(x) = x^{p/q} = (x^{1/q})^p$

$$f'(x) \overset{\text{chain}}{=} p\,(x^{1/q})^{p-1}\underbrace{\frac{1}{q}x^{\frac{1}{q}-1}}_{\text{inverse}} = \frac{p}{q}x^{p/q-1}$$

Def Let $f: X \to \mathbb{R}$, $x \in \mathrm{Dom}(f)$.

We say

- f has local maximum at x if $\exists \delta > 0\ \forall y \in B(x,\delta) \cap \mathrm{Dom}(f)$: $f(y) \le f(x)$.
- f has local minimum at x if $\exists \delta > 0\ \forall y \in B(x,\delta) \cap \mathrm{Dom}(f)$: $f(x) \le f(y)$

local extremum = local min or local max.

Thm (1st deri

Let $f: \mathbb{R} \to$

Then f has local

Assume f ha

Pf: Let $\delta > 0$ s

Then $\forall z \in$

$z > x \Rightarrow$

So f'

$z < x \Rightarrow$

Now $f'(x)$ e

...rse $f^{-1}(x) = x^{1/2}$

(x)

$\ldots(x))^{2-1} = \frac{1}{2(x^{1/2})^{2-1}} = \frac{1}{2}x^{1/2-1}$

$= (x^{1/2})^p$

$\ldots^{p-1}\frac{1}{2}x^{\frac{1}{2}-1} = \frac{p}{2}x^{p/2-1}$

Df Let $f: X \to \mathbb{R}$, $x \in \mathrm{Dom}(f)$.

We say

- f has local maximum at x

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  $f(y) \le f(x)$.

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  $f(x) \le f(y)$

local extremum = local min or loc. max.

Thm (1st derivative test).

Let $f: \mathbb{R} \to \mathbb{R}$, $x \in \underline{\mathrm{int}}\,\mathrm{Dom}(f)$. Assume $f'(x)$ exists.

Then f has local extremum at $x \Rightarrow f'(x) = 0$

Pf: Assume f has local max at x. Let $\delta > 0$ s.t. $(x-\delta, x+\delta) \subseteq \mathrm{Dom}(f)$.

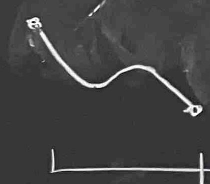
Then $\forall z \in (x-\delta, x+\delta)$:

$$z > x \Rightarrow \frac{f(z)-f(x)}{z-x} \le 0$$

So $f'_{+}(x) \le 0$. Similarly

$$z < x \Rightarrow \frac{f(z)-f(x)}{z-x} \ge 0 \Rightarrow f'_{-}(x) \ge 0$$

Now $f'(x)$ exists $\Rightarrow f'_{+}(x) = f'_{-}(x)$ so $f'(x) = 0$ $\boxtimes$



Mean value theorems

Thm (MVT) Let $a<b$, $f:[a,b]\to\mathbb{R}$ ($\mathrm{Dom}(f)=[a,b]$) s.t. f cont. on $[a,b]$ and differentiable on (a,b).

Then

(1) (Rolle's Thm) $f(b)=f(a) \Rightarrow \exists x\in(a,b): f'(x)=0$

(2) (Lagrange's Thm) $\exists x\in(a,b): f'(x)=\dfrac{f(b)-f(a)}{b-a}$

(3) (Cauchy's Thm) Let $g:[a,b]\to\mathbb{R}$ be cont. on $[a,b]$ and diff. on (a,b). Assume $\forall x\in(a,b): g'(x)\neq 0$.

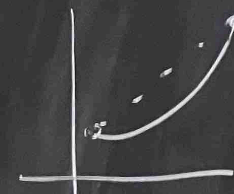
Then $g(a)\neq g(b)$ and

$$\exists x\in(a,b): \frac{f'(x)}{g'(x)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

Pf (1): $f(a)=f(b) \Rightarrow \left\langle \begin{array}{l} f \text{ constant} \\ f \text{ NOT constant} \end{array} \right\} \exists\, x \in (a,b):$ f has loc. extremum at x

Since f achieves its min & max

So $\exists x\in(a,b): f'(x)=0$



(2) $$h(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$$

Then $h(a)=f(a)$, $h(b)=f(a)$

Rolle's Thm: $\exists x\in(a,b): h'(x)=0 \Rightarrow f'(x) - \frac{f(b)-f(a)}{b-a} = 0$

(3) Lagrange's Thm: $\forall x\in(a,b): g'(x)\neq 0 \Rightarrow g(b)\neq g(a)$.

So define: $$h(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)}\left(g(x)-g(a)\right)$$

Then $h(a)=h(b)$ so Rolle's Thm: $\exists x\in(a,b): h'(x)=0$.

Now do the algebra. $\boxtimes$

Applications of MVT

Lemma Let $a<b$, $f:[a,b]\to\mathbb{R}$ cont on $[a,b]$, differentiable on (a,b). Then

$$f \text{ non-decreasing on } [a,b] \iff \forall x\in(a,b): f'(x)\ge 0$$

Pf $\Rightarrow$ triv (def. of derivative)

$\Leftarrow$ $\forall x,y\in[a,b]$:

$$x<y \underset{MVT}{\Rightarrow} \exists z\in(x,y): \underbrace{f(y)-f(x)}_{\ge 0} = \underbrace{f'(z)}_{\ge 0}\underbrace{(y-x)}_{\ge 0} \qquad \boxtimes$$

Corollary Let $f,g:[a,b]\to\mathbb{R}$ be cont. on $[a,b]$, diff. on (a,b).

Then

$$f(a)\le g(a) \wedge \forall x\in(a,b): f'(x)\le g'(x)$$

implies

$$\forall x\in[a,b]: f(x)\le g(x)$$

Pf: Apply MVT to $h=g-f$.

$$0=h(z) \Rightarrow \exists x\in(a,z): h'(z)=\frac{h(z)-h(a)}{z-a}<0$$

Ex Solve ODE:

Suppose $z(0)<y$

implies $\forall t\ge 0$

More abstract applicat

Def $f:[a,b]\to\mathbb{R}$

value property (

$\forall x,y\in[a,b]$:

...VT

$f: [a,b] \to \mathbb{R}$

...tiable on (a,b). Then

... on $[a,b]$

...(a,b): $f'(x) \geq 0$

...of derivative)

...$b]$:

$z \in (x,y)$: $f(y) - f(x) = \underbrace{f'(z)}_{\geq 0}\underbrace{(y-x)}_{\geq 0}$

≥ 0 ☒

<u>Corollary</u> Let $f,g: [a,b] \to \mathbb{R}$ be cont. on $[a,b]$, diff. on (a,b).

Then

$f(a) \leq g(a) \wedge \forall x \in (a,b): f'(x) \leq g'(x)$

implies

$\forall x \in [a,b]: f(x) \leq g(x)$

<u>Pf</u>: Apply MVT to $h = g - f$

$0 = h(z) \Rightarrow \exists x \in (a,z): h'(z) = \dfrac{h(z) - h(a)}{z - a} < 0$

<u>Ex</u> Solve ODE: $\dfrac{dy}{dt} = y + \sqrt{y}$

$\dfrac{dz}{dt} = z \quad \ldots \quad z = Ae^t$

Suppose $z(0) < y(0)$. Then Corollary

implies: $\forall t \geq 0$: $z(t) \leq y(t)$. (differential inequality)

More abstract application/explanation of MVT:

<u>Def</u> $f: [a,b] \to \mathbb{R}$ has <u>intermediate value property</u> (IVP) if

$\forall x,y \in [a,b]$: $x < y \Rightarrow f([x,y])$ <u>is an interval</u>

__Thm__ (Darboux's Thm) Let $f : [a,b] \to \mathbb{R}$ be cont on $[a,b]$, diff. on (a,b). Then f' has IVP on (a,b).

__Pf__: Let $x, y \in (a,b)$, WLOG: $f'(x) < f'(y)$. Pick $t \in (f'(x), f'(y))$. Define $h(z) = f(z) - tz$. Then $h'(x) = f'(x) - t < 0$ and $h'(y) = f'(y) - t > 0$. So h achieves its minimum inside (x,y), say at $z \in (x,y)$. Then $h'(z) = 0 \Rightarrow f'(z) = t$. $\boxtimes$