$\mathcal{F}$ = fermé = closed

Last time:

Thm Let $f: X \to Y$, $\mathrm{Dom}(f) = X$. Then
$\{x \in X : f$ discontinious at $x\}$ is $\mathcal{F}_\sigma$-set

$\mathcal{F}_\sigma$ set A = a countable union of closed sets
$\mathcal{G}_\delta$ set A = a contable intersection of open sets.

Fact: $A \in \mathcal{F}_\sigma \iff X \setminus A \in \mathcal{G}_\delta$

Thm $X \setminus \mathbb{Q}$ is NOT $\mathcal{F}_\sigma$-set

Cor No $f: \mathbb{R} \to \mathbb{R}$ has $\{x \in \mathbb{R} : f$ discontinuous$\} = X \setminus \mathbb{Q}$.

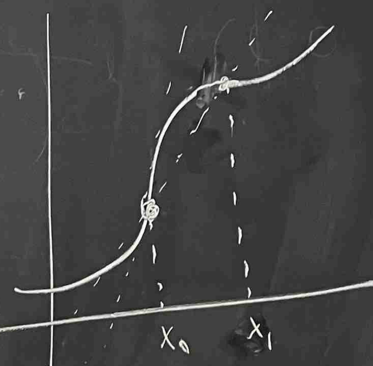
Lemma: Let $A \subseteq \mathbb{R}$ be an $\mathcal{F}_\sigma$-set. Then $\exists f : \mathbb{R} \to \mathbb{R}$, $\mathrm{Dom}(f) = \mathbb{R}$ s.t.

$$\{x \in \mathbb{R} : f \text{ discontinuous at } x\} = A$$

The derivative

Secant line:

$$y = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + f(x_0)$$



Def Let $f : \mathbb{R} \to \mathbb{R}$, $x \in \mathrm{int}\,\mathrm{Dom}(f)$. We say that f is differentiable at x if the limit

$$f'(x) := \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

exists. We call $f'(x)$ the derivative of f at x

- $f'(x)$ Langrange's notation
- $\frac{df}{dx}(x)$ Leibnitz notation

Alternative formula

$$f'(x) := \lim_{h \to 0}$$

One sided derivat

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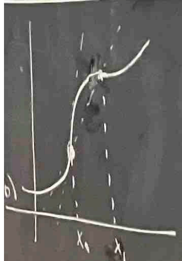

Def Let $f:\mathbb{R}\to\mathbb{R}$, $x\in \operatorname{int} Dom(f)$.
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- $f'(x)$ Lagrange's notation
- $\frac{df}{dx}(x)$ Leibnitz notation

Alternative formula:

$$f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

One sided derivatives:

Def If $\exists \delta>0$ s.t. $[x, x+\delta) \subseteq Dom(f)$, we set

$$f'_+(x) = \frac{df}{dx^+}(x) = \lim_{z\to x^+} \frac{f(z)-f(x)}{z-x}$$

and call this the <u>right derivative of f at x</u>

Similarly, if $\exists \delta>0$: $(x-\delta, x] \subseteq Dom(f)$, then

$$f'_-(x) = \frac{df}{dx^-}(x) = \lim_{z\to x^-} \frac{f(z)-f(x)}{z-x}$$

is the <u>left derivative</u> of f at x

Lemma $\forall x \in \operatorname{int} \operatorname{Dom}(f)$

$$f'(x) \text{ exists} \iff f'^{,+}(x),\, f'^{,-}(x) \text{ exist} \;\wedge\; f'^{,+}(x) = f'^{,-}(x)$$

Computing $f'(x)$:

- $f(x) = \text{constant} \Rightarrow \forall x : f'(x) = 0$
- $f(x) = ax + b \Rightarrow \forall x : f'(x) = a$

- $f(x) = x^2$

$$\frac{f(z)-f(x)}{z-x} = \frac{z^2-x^2}{z-x} = \frac{(z-x)(z+x)}{z-x} = z+x$$

So $f'(x) = 2x$

- $f(x) = x^n \quad (n \in \mathbb{N})$

$$f(z)-f(x) = z^n - x^n = (z-x)\sum_{k=0}^{n-1} z^k x^{n-1-k}$$

So $$\frac{f(z)-f(x)}{z-x} = \sum_{k=0}^{n-1} z^k x^{n-1-k} \xrightarrow[z \to x]{} n x^{n-1}$$

So $f'(x) = n x^{n-1}$

Lemma Let $\alpha \in \mathbb{R}$, let $f : (0,\infty) \to \mathbb{R}$ be defied by $f(x) = x^{\alpha}$. Then f is differentiable on $(0,\infty)$, and

$$\forall x > 0 : \quad f'(x) = \alpha x^{\alpha - 1}$$

Tomorrow: proof for $\alpha \in \mathbb{Q}$.

For $\alpha \notin \mathbb{Q}$, proof via exponential function later.

Linear approximation

First note that differentiability is a statement of regularity

Lemma Let $x \in \operatorname{int} \operatorname{Dom}(f)$. Then

$$\boxed{f'(x) \text{ exists} \Rightarrow f \text{ continuous at } x}$$

Pf: Assume $f'(x)$ exists and set

$$u_x(z) := \begin{cases} \frac{f(z)-f(x)}{z-x} - f'(x) & \text{if } z \neq x \\ 0 & \text{if } z = x \end{cases}$$

$$f'(x) \text{ exists} \Rightarrow \lim_{z \to x} u_x(z) = 0 = u_x(x)$$

$\Leftrightarrow u_x$ continuous at x

Note $\forall z \in \operatorname{Dom}(f) = \operatorname{Dom}($

$f(z) - f(x) = [f$

Now u_x continuous at

$\exists \delta > 0 \ \forall z \in \operatorname{Dom}(f) :$

$|z - x| < \delta \Rightarrow |u_x($

$\Rightarrow |f(z)$

So f is even locally Lipsc

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$$\Leftrightarrow u_x \text{ continuous at } x$$

Note $\forall z \in \operatorname{Dom}(f) = \operatorname{Dom}(u_x)$

$$f(z) - f(x) = \left[f'(x) + u_x(z) \right] (z - x)$$

Now u_x continuous at x means:

$$\exists \delta > 0 \ \forall z \in \operatorname{Dom}(f):$$
$$|z - x| < \delta \Rightarrow |u_x(z)| \leq 1$$
$$\Rightarrow |f(z) - f(x)| \leq \left[1 + |f'(x)| \right] |z - x|$$

So f is even locally Lipschitz continuous at x. $\boxtimes$

<u>Lemma</u>: $\forall\, x \in \operatorname{int} \operatorname{Dom}(f)$:

$f'(x)$ exists

$$\iff \exists\, a \in \mathbb{R} : \inf_{\delta > 0} \sup_{\substack{z \in \operatorname{Dom}(f) \\ |z - x| < \delta}} \frac{1}{\delta} \left| f(z) - f(x) - a(z - x) \right| = 0$$

(Linear approximation)

<u>Note</u> Such $a \in \mathbb{R}$, if exists, is necessarily unique.

Lemma (Sum & Product rule) Let $f, g : \mathbb{R} \to \mathbb{R}$ be s.t. $f'(x), g'(x)$ exist. Then $(f+g)'(x)$, $(f \cdot g)'(x)$ exist and

$$(f+g)'(x) = f'(x) + g'(x)$$

$$(f \cdot g)'(x) = f'(x) g(x) + g'(x) f(x)$$

Pf of Product: $\dfrac{(f \cdot g)(z) - (f \cdot g)(x)}{z - x} = \left[\dfrac{f(z) - f(x)}{z-x}\right] g(z) + \left[\dfrac{g(z) - g(x)}{z - x}\right] f(x)$

Take limits via product/sum rule for limits and the continuity of g at x. $\boxtimes$

Lemma (Chain Rule)

Let $f, g : \mathbb{R} \to \mathbb{R}$, $x \in \operatorname{int} \operatorname{Dom}(f)$ s.t. $f(x) \in \operatorname{int} \operatorname{Dom}(g)$. Assume $f'(x)$ and $g'(f(x))$ exists. Then $g \circ f$ is differentiable at x and

$$(g \circ f)'(x) = g'(f(x)) f'(x)$$

Pf $f'(x)$ exists gives:

$$f(z) - f(x) = [f'(x) + u_x(z)](z - x)$$

$g'(f(x))$ exists gives:

corresponds to g

$$g(y) - g(f(x)) = [g'(f(x)) + v_{f(x)}(y)](y - f(x))$$

So

$$g(f(z)) - g(f(x)) = [g'(f(x)) + v_{f(x)}(f(z))][f'(x) + u_z(x)](z - x)$$

Now recall:

$$\lim_{z \to x} u_x(z) = 0$$

$$\lim_{y \to f(x)} v_{f(x)}(y)$$

Cont. of f at x

$$\Rightarrow \lim_{z \to x} f(z) =$$

So $\lim_{z \to x} \frac{g(f(z)) - g(f(x))}{z - x}$

...in Rule)

... $\mathbb{R}\to\mathbb{R}$, $x \in \operatorname{int} \operatorname{Dom}(f)$

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Pf $f'(x)$ exists gives:

$$f(z)-f(x) = [f'(x)+u_x(z)](z-x)$$

$g'(f(x))$ exists gives:

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$$g(y)-g(f(x)) = \left[g'(f(x)) + v_{f(x)}(y)\right](y-f(x))$$

So $g(f(z))-g(f(x))$

$$= \left[g'(f(x))+v_{f(x)}(f(z))\right]\left[f'(x)+u_z(x)\right](z-x)$$

Now recall:

$$\lim_{z\to x} u_x(z) = 0$$

$$\lim_{y\to f(x)} v_{f(x)}(y) = 0$$

Cont. of f at x

$$\Rightarrow \lim_{z\to x} f(z) = f(x)$$

So $$\lim_{z\to x} \frac{g(f(z))-g(f(x))}{z-x} = g'(f(x))f'(x)$$

$\boxtimes$