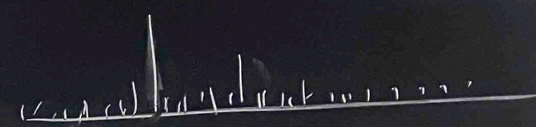
Discontinuities

Ex $f : \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = \begin{cases} \frac{1}{n+1} & x = q_n \\ 0 & x \in \mathbb{Q} \end{cases}$ ... $\mathbb{Q} = \{q_n\}_{n \in \mathbb{N}}$ enumeration of $\mathbb{Q}$

$\forall x \in \mathbb{R}: \lim_{z \to x} f(z) = 0$



f continuous on $\mathbb{R} \setminus \mathbb{Q}$

f NOT continuous on $\mathbb{Q}$

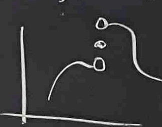
Def $f : X \to Y$ has removable discontinuity at x if $\lim_{z \to x} f(z)$ exists.

Def Let $f : \mathbb{R} \to Y$. We say that f has

- discontinuity of 1^{st} kind at x

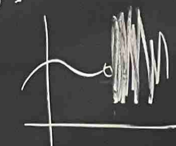
  if $f(x^+), f(x^-)$ exist

  yet $|\{f(x^+), f(x), f(x^-)\}| \geq 2$



- discontinuity of 2^{nd} kind

  if one or both of $f(x^+), f(x^-)$ don't exist



Ex $f(x) = \mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

has disc. of 2^{nd} kind at every $x \in \mathbb{R}$

Def Let $f : \mathbb{R} \to \mathbb{R}$ $(\mathrm{Dom}(f) = \mathbb{R})$ be monotone. Then

$\forall x \in \mathbb{R} : f(x^+), f(x^-)$ exist

Pf: Let $f : \mathbb{R} \to \mathbb{R}$ be non decreasing.

claim $f(x^+) = \inf\{f(z) : z > x\}$

Pf Denote infimum as c. Then

$\forall \varepsilon > 0 \; \exists \tilde{z} > x : c \leq f(\tilde{z}) < c + \varepsilon$

Now $f \uparrow \Rightarrow \forall z \in (x, \tilde{z}) : c \leq f(z) \leq c + \varepsilon$

Take $\delta := \tilde{z} - x$. Then

$\forall z \in (x, x+\delta) : |f(z) - c| < \varepsilon$

So $\lim_{z \to x^+} f(z) = c$ $\boxtimes$

Lemma Let $f : \mathbb{R}$

Then

$x \mapsto f(x^+)$ is

$x \mapsto f(x^-)$ is

Moreover:

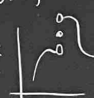
$\forall x \in \mathbb{R} : f($

$\forall x < y : f($

Pf HW

... We say that f has ...

... kind at x



... (x^-) exist

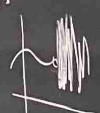
..., $f(x), f(x^-)\}| \geq 2$

... 2nd kind

... $f(x^+), f(x^-)$ don't exist

... $x \in \mathbb{Q}$

$x \notin \mathbb{Q}$



... at every $x \in \mathbb{R}$

Def Let $f: \mathbb{R} \to \mathbb{R}$ $(\mathrm{Dom}(f) = \mathbb{R})$ be monotone. Then

$$\forall x \in \mathbb{R}: \ f(x^+), f(x^-) \text{ exist}$$

Pf: Let $f: \mathbb{R} \to \mathbb{R}$ be non decreasing.

claim $f(x^+) = \inf\{f(z) : z > x\}$

Pf Denote infimum as c. Then

$\forall \varepsilon > 0 \ \exists \tilde{z} > x: \ c \leq f(\tilde{z}) < c + \varepsilon$

Now $f \uparrow \Rightarrow \forall z \in (x, \tilde{z}): \ c \leq f(z) < c + \varepsilon$

Take $\delta := \tilde{z} - x$. Then

$\forall z \in (x, x+\delta): \ |f(z) - c| < \varepsilon$

So $\lim_{z \to x^+} f(z) = c$. $\boxtimes$

Lemma Let $f: \mathbb{R} \to \mathbb{R}$ be non-decreasing.

Then $x \mapsto f(x^+)$ is right continuous

$x \mapsto f(x^-)$ is left continuous

Moreover:

$$\forall x \in \mathbb{R}: \ f(x^-) \leq f(x) \leq f(x^+)$$

$$\forall x < y: \ f(x^+) \leq f(y^-)$$

Pf HW

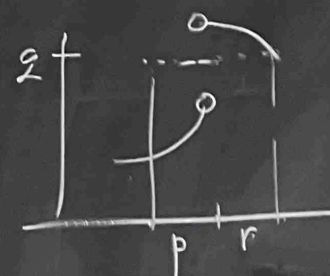
Lemma Let $f : \mathbb{R} \to \mathbb{R}$ be monotone. Then
$\{x \in \mathbb{R} : f(x^+) \neq f(x^-)\}$ is finite or countable

Pf: Assume f non-decreasing.
Denote $A := \{x \in \mathbb{R} : f(x^+) \neq f(x^-)\}$
Enumerate $\mathbb{Q}$ into $\{q_n\}_{n \in \mathbb{N}}$. Define $\sigma : A \to \mathbb{N}$
$$\sigma(x) := \inf\{n \in \mathbb{N} : f(x^-) < q_n < f(x^+)\}$$
Note: $f(x^-) < q_{\sigma(x)} < f(x^+)$
So $x, y \in A$: $x < y \Rightarrow q_{\sigma(x)} < f(x^+) \leq f(y^-) < q_{\sigma(y)}$.
So $x \neq y \Rightarrow \sigma(x) \neq \sigma(y)$ so σ is injective.
$\boxtimes$

Generalizes to:

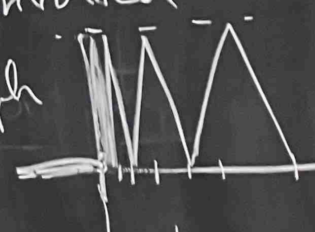
<u>Lemma</u> Let $f : \mathbb{R} \to \mathbb{R}$ be s.t. $\forall x \in \mathbb{R}$: $f(x^+), f(x^-)$ exist

Then $\{ x \in \mathbb{R} : f(x^+) \neq f(x^-) \}$ is at most countable

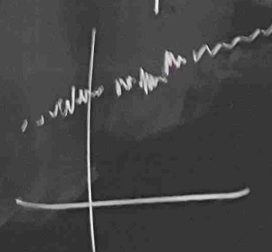


<u>Note</u>: • $1_{\mathbb{Q}}$ shows this fails when discontinuities of 2^{nd} get involved

• Let h = function with graph



Define $f(x) = \sum_{n=0}^{\infty} 2^{-n} \cdot h(x - q_n)$ — q_n enumerates $\mathbb{Q}$.



has discontinuity of 2^{nd} kind at every $x \in \mathbb{Q}$.

Q: How bad the set of discontinuities can be?

Thm Let $f : X \to Y$ ($\mathrm{Dom}(f) = X$),
Then $\{ x \in X : f$ NOT continuous at $x \}$ is a countable union of closed sets.

Pf Define
$$\mathrm{osc}_f(x) := \inf_{\delta > 0} \sup \{ \rho(f(z), f(y)) : z, y \in B_X(x, \delta) \}$$

claim f continuous at $x \iff \mathrm{osc}_f(x) = 0$

Pf of claim:

f cont at $x \Rightarrow$ given $\varepsilon > 0$ $\exists \delta > 0$
$\forall y \in B_X(x, \delta) : \rho(f(x), f(y)) < \varepsilon$

$\Rightarrow \forall \varepsilon > 0\ \exists \delta > 0\ \forall z, y \in B_X(x, \delta)$:
$\rho(f(z), f(y)) \le \rho(f(x), f(z)) + \rho(f(x), f(y))$
$< \varepsilon + \varepsilon = 2\varepsilon$

$\Rightarrow \sup \{ \rho(f(z), f(y)) : y, z \in B(x, \delta) \} \le 2\varepsilon$

Conversely: $\mathrm{osc}_f(x) = 0 \Rightarrow \forall \varepsilon > 0\ \exists \delta > 0$:
$\sup \{ \rho(f(z), f(y)) : y, z \in B(x, \delta) \} < \varepsilon$
$\Rightarrow \forall z \in B(x, \delta) : \rho(f(z), f(x)) < \varepsilon$

$\forall \varepsilon > 0$:
claim $\{ x \in X :$

Pf: Let $x \in X$
Then $\exists \delta > 0$
If $\tilde{x} \in B$
Then
$\mathrm{osc}_f(\tilde{x})$

So $B(x, \delta$

Now $\{ x \in X : f$ NOT cont
$= \{ x \in X :$

...t of discontinuities,

$(\mathrm{Dom}(f) = X)$,

NOT continuous at $x\}$

...nion of closed sets.

...$\sup\{\rho(f(z), f(y)) : z, y \in B_X(x,\delta)\}$

...uous at $x \iff \mathrm{osc}_f(x) = 0$

Pf of claim:

f cont at $x \Rightarrow$ given $\varepsilon > 0$ $\exists \delta > 0$ $\forall y \in B_X(x,\delta) : \rho(f(x), f(y)) < \varepsilon$

$\Rightarrow \forall \varepsilon > 0\ \exists \delta > 0\ \forall z, y \in B_X(x,\delta)$;

$\rho(f(z), f(y)) \le \rho(f(x), f(z)) + \rho(f(x), f(y)) < \varepsilon + \varepsilon = 2\varepsilon$

$\Rightarrow \sup\{\rho(f(z), f(y)) : y, z \in B(x,\delta)\} \le 2\varepsilon$

Conversely: $\mathrm{osc}_f(x) = 0 \Rightarrow \forall \varepsilon > 0\ \exists \delta > 0$:

$\sup\{\rho(f(z), f(y)) : y, z \in B(x,\delta)\} < \varepsilon$

$\Rightarrow \forall z \in B(x,\delta) : \rho(f(z), f(x)) < \varepsilon$

claim: $\forall \varepsilon > 0$: $\{x \in X : \mathrm{osc}_f(x) < \varepsilon\}$ is open

Pf: Let $x \in X$ be s.t. $\mathrm{osc}_f(x) < \varepsilon$.

Then $\exists \delta > 0$: $\sup\{\rho(f(y), f(z)) : y, z \in B(x,\delta)\} < \varepsilon$.

If $\tilde{x} \in B(x, \delta/2)$, then $B(\tilde{x}, \delta/2) \subseteq B(x, \delta)$

Then $\mathrm{osc}_f(\tilde{x}) \le \sup\{\rho(f(y), f(z)) : y, z \in B(\tilde{x}, \delta/2)\} < \varepsilon$

So $B(x, \delta/2) \subseteq \{y \in X : \mathrm{osc}_f(y) < \varepsilon\}$. $\boxtimes$

Now $\{x \in X : f \text{ NOT continuous at } x\}$

$= \{x \in X : \mathrm{osc}_f(x) > 0\} = \bigcup_{n \in \mathbb{N}} \underbrace{\{x \in X : \mathrm{osc}_f(x) \ge 2^{-n}\}}_{\text{closed}}$

Thm $\mathbb{Q}$ is NOT a countable intersection of open sets (Baire category theorem)

Pf Assume $\mathbb{Q} = \bigcap_{n\in\mathbb{N}} U_n$ where $\{U_n\}_{n\in\mathbb{N}}$ are open sets.

WLOG: U_n decreasing. Define $\{n_k\}$, $\{\delta_k\}$ as follows:

$\mathbb{Q} = \{q_n\}_{n\in\mathbb{N}}$.

$n_0 = 0$, $\delta_0 = \sup\{\delta \in (0,1) : [q_0 - 2\delta, q_0 + 2\delta] \subseteq U_0\}$

$n_{k+1} := \inf\{n > n_k : q_n \in [q_{n_k} - \delta_k, q_{n_k} + \delta_k]\}$

$\delta_{k+1} := \sup\{\delta \in (0,1) : [q_{n_{k+1}} - 2\delta, q_{n_{k+1}} - 2\delta] \subseteq [q_{n_k} - \delta_k, q_{n_k} + \delta_k] \cap U_{k+1}$

$\wedge \forall j \le n_k : |q_j - q_{n_{k+1}}| \geq 2\delta\}$

Denote $I_k = [q_{n_k} - \delta_k, q_{n_k} + \delta_k]$.

Then $\{I_k\}_k$ are closed, nested so $\exists\, x \in \bigcap_{k=0}^{\infty} I_k$.

yet $\forall j = 0, \dots, n_k : q_j \notin I_k$ so $x \notin \{q_n : n \in \mathbb{N}\} = \mathbb{Q}$.

Contradiction because $I_k \subseteq U_k$ gives

$$\bigcap_{k=0}^{\infty} I_k \subseteq \bigcap_{k=0}^{\infty} U_k = \mathbb{Q}. \quad \text{so } x \in \mathbb{Q} \quad !$$