Limits & discontinuities

Def Let $f: X \to Y$ be function between metric spaces X, Y.
Let $x \in \mathrm{Dom}(f)$ be NOT isolated. Let $y \in Y$.
We say that "f has limit y at x" or "y is limit of f at x"
if $\forall \varepsilon > 0\ \exists \delta > 0\ \forall z \in \mathrm{Dom}(f) \setminus \{x\}$:

$$0 < \rho_X(x,z) < \delta \implies \rho_Y(f(z), y) < \varepsilon$$

Note $f(x)$ immaterial for the limit!

Lemma Let $y, \tilde{y}$ be limits of f at x (for $x \in \overline{\mathrm{Dom}(f)}$ NOT isolated). Then

$$y = \tilde{y}$$

Pf Let $\varepsilon := \frac{1}{2}\rho(y,\tilde{y})$. Assume $\varepsilon > 0$.

x NOT isolated $\Rightarrow \exists \delta > 0 \ \exists z \in B_X(x,\delta)$

$$z \neq x \wedge \rho_Y(f(z),y) < \varepsilon \wedge \rho_Y(f(z),\tilde{y}) < \varepsilon$$

But then

$$\rho(y,\tilde{y}) \leq \rho(f(z),y) + \rho(f(z),\tilde{y}) < 2\varepsilon = \rho(y,\tilde{y})$$

Absurd $\Rightarrow \varepsilon = 0$. !

Notation $\lim_{z\to x} f(z) := y$

$f(z) \xrightarrow[z \to x]{} y$

Lemma For $x \in \mathrm{Dom}(f)$ NOT isolated:

f continuous at $x \iff \lim_{z\to x} f(z) = f(x)$

Pf Add $z = x$ to def. of limit.

Lemma Let $x \in \overline{\mathrm{Dom}(f)}$ NOT isolated. For $y \in Y$

set $g(z) := \begin{cases} f(z) & z \neq x \\ y & z = x \end{cases}$. Then

$\lim_{z\to x} f(z) = y \iff g$ is continuous at x

Thm (AC) (Sequen

Let $x \in \overline{\mathrm{Dom}(f)}$ NO

Then

$\lim_{z\to x} f(z) = y \iff \forall$

Pf $\Rightarrow$ Suppose $\lim_{z\to x} f$

Def of limit: Given $\varepsilon > 0$

& $\rho_X(x,z) < \delta \Rightarrow \rho_Y$

$z_n \to x$, $z_n \neq x \Rightarrow \exists n_0 \in \mathbb{N} \ \forall n \geq n_0$:

So $\forall n \geq n_0$, $\rho_Y(f(z_n), y$

...nts of f at x
(NOT isolated). Then

$\}$. Assume $\varepsilon>0$.
$\Rightarrow \exists \varepsilon>0\ \exists z \in B_X(x,\delta)$
$\rho_Y(f(z),y)<\varepsilon \wedge \rho_Y(f(z),\tilde{y})<\varepsilon$

$\rho(\cdot)+\rho(f(z),\tilde{y})<2\varepsilon=\rho(y,\tilde{y})$
!

Notation $\lim_{z\to x} f(z)=y$

$f(z)\xrightarrow[z\to x]{} y$

Lemma For $x\in \mathrm{Dom}(f)$ NOT isolated:

f continuous at $x \Leftrightarrow \lim_{z\to x} f(z)=f(x)$

Pf Add $z=x$ to def. of limit.

Lemma Let $x\in\overline{\mathrm{Dom}(f)}$ NOT isolated. For $y\in Y$ set $g(z):=\begin{cases} f(z) & z\neq x\\ y & z=x\end{cases}$. Then

$\lim_{z\to x} f(z)=y \Leftrightarrow g$ is continuous at x

Thm (AC) (Sequential characterization)
Let $x\in\overline{\mathrm{Dom}(f)}$ NOT isolated. Let $y\in Y$.
Then
$$\lim_{z\to x} f(z)=y \Leftrightarrow \forall \{z_n\}_{n\in\mathbb{N}}\in(\mathrm{Dom}(f)\setminus\{x\})^{\mathbb{N}}:$$
$$z_n\to x \Rightarrow f(z_n)\to y$$

$\boxed{\text{Pf} \Rightarrow}$ Suppose $\lim_{z\to x} f(z)=y$. Let $\{z_n\}_{n\in\mathbb{N}}$ be s.t.
$z_n\to x \wedge \forall n\in\mathbb{N}: z_n\neq x$

Def of limit: Given $\varepsilon>0\ \exists\delta>0$:
$0<\rho_X(x,z)<\delta \Rightarrow \rho_Y(f(z),y)<\varepsilon$

$\begin{matrix} z_n\to x\\ z_n\neq x\end{matrix} \Rightarrow \exists n_0\in\mathbb{N}\ \forall n\geq n_0: 0<\rho(z_n,x)<\delta$.

So $\forall n\geq n_0$: $\rho_Y(f(z_n),y)<\varepsilon$. So $f(z_n)\to y$.

Pf of $\Leftarrow$ Assume $\neg\left(\lim_{z\to x} f(z) = y\right)$.

So $\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists z \in \mathrm{Dom}(f)$:

$$0 < \rho_X(x, z) < \delta \ \wedge \ \rho_Y(f(z), y) \geq \varepsilon$$

So $\forall n \in \mathbb{N} \ \exists z_n \in B_X(x, 2^{-n}) \setminus \{x\} : \ \rho_Y(f(z_n), y) \geq \varepsilon$

(AC needed to have these as $\{z_n\}_{n\in\mathbb{N}}$)

Now $z_n \to x$ yet $\rho(f(z_n), y) \geq \varepsilon$

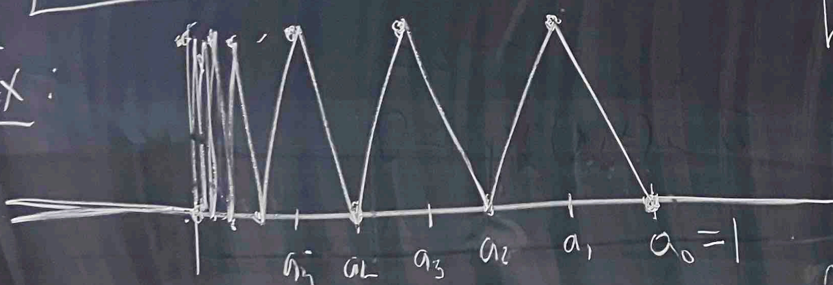
So $\neg(f(z_n) \to y)$. $\boxtimes$

Ex Dirichlet function:

$$f(z) = \mathbb{1}_{\mathbb{Q}}(z) := \begin{cases} 1 & z \in \mathbb{Q} \\ 0 & z \notin \mathbb{Q} \end{cases}$$

$\lim_{z \to x} f(z)$ Does NOT exist at any $x \in \mathbb{R}$

Ex:



$$h(x) = \begin{cases} 0 & x \in \{a_{2n} : n \in \mathbb{N}\} \\ 1 & x \in \{a_{2n+1} : n \in \mathbb{N}\} \\ \text{linear} & \text{else (for } x \geq 0) \\ 0 & x < 0 \\ & x > a_0 \end{cases}$$

$\lim_{x \to 0} h(x)$ does NOT exist.

Lemma ("Rules" for limits) Let $f, g : X \to \mathbb{R}$.

Then

$$\lim_{z \to x} [f(z) + g(z)] = \lim_{z \to x} f(z) + \lim_{z \to x} g(z)$$

$$\lim_{z \to x} [f(z) \cdot g(z)] = \left(\lim_{z \to x} f(z)\right) \cdot \left(\lim_{z \to x} g(z)\right)$$

and if $\lim_{z \to x} g(z) \neq 0$, then

$$\lim_{z \to x} \frac{f(z)}{g(z)} = \frac{\lim_{z \to x} f(z)}{\lim_{z \to x} g(z)}$$

provided $\lim_{z \to x} f(z)$, $\lim_{z \to x} g(z)$ exist (in $\mathbb{R}$)

Def (Limit from a subset)

Let $A \subseteq \mathrm{Dom}(f)$, $x \in \overline{A}$ NOT isolated.

Then $\lim_{\substack{z \to x \\ z \in A}} f(z) := \lim_{z \to x} f_A(z)$

where $f_A = f|_A$.

Def Let $f : \mathbb{R} \to Y$. Let $x \in \overline{\mathrm{Dom}(f)}$.

If x NOT isolated in $\overline{\mathrm{Dom}(f) \cap [x, \infty)}$:

$$\lim_{z \to x^+} f(z) := \lim_{\substack{z \to x \\ z \in \mathrm{Dom}(f) \cap [x,\infty)}} f(z)$$

Similarly for $\lim_{z \to x^-} f(z)$.

Nota...

Lemma

Then

lim

If HW

Def (Limit from a subset)

Let $A \subseteq \overline{\mathrm{Dom}(f)}$, $x \in \overline{A}$ NOT isolated.

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Def Let $f: \mathbb{R} \to Y$. Let $x \in \overline{\mathrm{Dom}(f)}$.

If x NOT isolated in $\overline{\mathrm{Dom}(f)} \cap [x, \infty)$:

$$\lim_{z \to x^+} f(z) := \lim_{\substack{z \to x \\ z \in \mathrm{Dom}(f) \cap (x, \infty)}} f(z)$$

Similarly for $\displaystyle\lim_{z \to x^-} f(z)$.

Notation: $\displaystyle f(x^+) := \lim_{z \to x^+} f(z)$

$\displaystyle f(x^-) := \lim_{z \to x^-} f(z)$

Lemma Let $f: \mathbb{R} \to Y$, $x \in \mathrm{int}\,\mathrm{Dom}(f)$.

Then $\forall y \in \mathbb{R}$:

$$\lim_{z \to x} f(z) = y \iff f(x^+), f(x^-) \text{ exist} \wedge f(x^+) = f(x^-) = y$$

Pf HW

Ex $\mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

$\left.\begin{array}{l} \mathrm{sgn}(0^+) = 1 \\ \mathrm{sgn}(0^-) = -1 \end{array}\right\}$ $\lim_{x \to 0} \mathrm{sgn}(x)$ does NOT exist.

Def $f : \mathbb{R} \to Y$ is

left continuous at x if $\lim_{z \to x^-} f(z) = f(x)$

right continuous at x if $\lim_{z \to x^+} f(z) = f(x)$