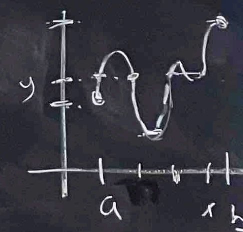
Intermediate Value Theorem

Thm (IVT, Bolzano 1817) Let $a<b$ be reals, $f:[a,b]\to\mathbb{R}$ continuous function with $\mathrm{Dom}(f)=[a,b]$.
Assume $f(a)\le f(b)$. Then

$$\forall y\in[f(a),f(b)]\ \exists x\in[a,b]:\ f(x)=y$$



Pf #1 WLOG: $f(a)<f(b)$, $y\in(f(a),f(b))$. Set

$$x := \sup\{z\in[a,b] : f(z)\le y\}$$

Note Def of sup + cont of f $\Rightarrow$ $f(x)\le y$

If $f(x)<y$ then $x\neq b$ then cont of $\Rightarrow$ $\exists\,\delta>0\ \forall x'\in[x,x+\delta): f(x')<y$.

then x is NOT the supremum. So $f(x)=y$.

Pf #2 **Idea** (Assuming $f(a)<f(b)$) check whether $y \le f\left(\frac{a+b}{2}\right)$ or $y > f\left(\frac{a+b}{2}\right)$

This leads to sequences $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$

s.t.

$$\forall n\in\mathbb{N}: \begin{cases} a_{n+1}=a_n \wedge b_{n+1} = \frac{a_n+b_n}{2} & \text{if } y \le f\left(\frac{a_n+b_n}{2}\right) \\ a_{n+1} = \frac{a_n+b_n}{2} \wedge b_{n+1}=b_n & \text{if } y > f\left(\frac{a_n+b_n}{2}\right) \end{cases}$$

Then we get:

$\forall n\in\mathbb{N}$: $a_{n+1} \ge a_n \wedge b_{n+1} \le b_n \wedge b_{n+1}-a_{n+1} = \frac{1}{2}(b_n-a_n)$

$\forall n\in\mathbb{N}$: $f(a_n) \le y \le f(b_n)$

Taking limits we get

$$x := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \in [a,b]$$

and by continuity of f at x:

$$f(x) \le y \le f(x)$$

which means $y=f(x)$. $\boxtimes$

Cor All odd-degree polynomials on $\mathbb{R}$ have a root in $\mathbb{R}$

Pf: $P(x) = x^{2n+1} + Q$...

Then $\lim_{x\to\pm\infty} \frac{Q(x)}{x^{2n+1}}$ ...

means $\exists M>0$:

$\forall x>M$: $P(x)$...

$\forall x<-M$: $P(x)$...

Since P is continuous ...

$\exists x\in[-M,M]$: ...

(b)) check
$y > f(\frac{a+b}{2})$

$\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$

$\frac{a_n+b_n}{2}$ if $y \le f(\frac{a_n+b_n}{2})$

$a_{n+1} = b_n$ if $y > f(\frac{a_n+b_n}{2})$

$a_{n+1} \le b_n \wedge b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$

$\le f(b_n)$

Taking limits we get

$$x := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \in [a,b]$$

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Cor All odd-degree polynomials on $\mathbb{R}$ have a root in $\mathbb{R}$

Pf: $P(x) = x^{2n+1} + Q(x)$ where $\deg(Q) \le 2n$

Then $\lim_{x\to\pm\infty} \frac{Q(x)}{x^{2n+1}} = 0$

meaning $\exists M > 0$:

$\forall x > M$: $P(x) > 0$

$\forall x < -M$: $P(x) < 0$

Since P is continuous, IVT:

$\exists x \in [-M, M]$: $P(x) = 0$. $\boxtimes$

Lemma Let $I \subseteq \mathbb{R}$ be bounded closed $(\neq \emptyset)$ interval

Let $f: I \to I$ be continuous will $\mathrm{Dom}(f) = I$.

Then $\exists x \in I : f(x) = x$

Pf (using IVT) HW

Note: I closed & $\mathrm{Dom}(f) = I$ crucial

Ex $I = (0,1)$, $f(x) = x/2$

Special case of:

Thm (Brower's fixed pt theorem)

Let $B = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$, $f: B \to B$ cont, $\mathrm{Dom}(f) = B$

Then $\exists x \in B : f(x) = x$

Cor: Let $f:\mathbb{R}\to\mathbb{R}$ be continuous & 2-periodic

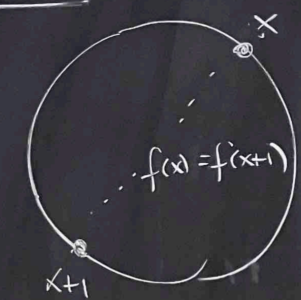
$(\forall x\in\mathbb{R}: f(x+2)=f(x))$

Then $\exists x\in\mathbb{R}: f(x+1)=f(x)$

Pf $g(x)=f(x+1)-f(x)$

Note g continuous

$$g(x+1)=f(x+2)-f(x+1)$$
$$= f(x)-f(x+1) = -g(x)$$



So one of the following is true:

$$g(0)=0=g(1) \vee g(0)<0<g(1) \vee g(0)>0>g(1)$$

So $\exists x\in[0,1]: g(x)=0$ $\boxtimes$

Def: A topological space X is **connected** if

$$\forall E \subseteq X : E \neq \emptyset \wedge E \text{ open} \wedge X \setminus E \text{ open} \implies E = X$$

Equivalent formulation:

$$\forall E, F \subseteq X : \left(E \text{ open} \wedge F \text{ open} \wedge E \cap F = \emptyset \wedge X = E \cup F \right) \implies E = \emptyset \vee F = \emptyset$$

What about subsets of X?

A: Use relative topology.

Def Given top. space $(X, \mathcal{T})$ and a subset $A \subseteq X$, the **relative topology** on A is

$$\mathcal{T}_A := \{ O \cap A : O \in \mathcal{T} \}$$

Ex $A = [0,1]$

set $[0, 1/2)$ is relatively open in A

Def A subset ... is **connected** ... in its relative ...

Ex of Not connecte...

$A = \mathbb{R} \setminus \{0\}$,

Lemma $\forall a < b$:

X is

$\wedge\ X \setminus E$ open

$= X$

open

$\wedge\ X = E \cup F$

$\emptyset \vee F = \emptyset$

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Def A subset $A \subseteq X$ of top. space X is connected ~~if~~ if A is connected in its relative topology

Ex of Not connected sets

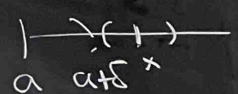
$A = \mathbb{R} \setminus \{0\}$, $\mathbb{R}^2 \setminus \mathbb{R}$, $\mathbb{R} \setminus (0,1) = (-\infty, 0] \cup [1, \infty)$

Lemma $\forall a < b$: $[a,b]$ is connected

Pf: Assume $E \subseteq [a,b]$ be rel. open, $E \neq \emptyset$

will $E^c := [a,b] \setminus E$ rel. open, $E^c \neq \emptyset$

WLOG: $a \notin E$.



Then $\exists \delta > 0$: $[a, a+\delta) \subseteq E^c$

Define $x = \inf(E)$. So $x \neq a$.

E rel. open $\Rightarrow$ $x \notin E$

But E^c rel. open $\Rightarrow$ $\exists \delta' > 0$:

$(x-\delta', x+\delta') \subseteq E^c$ so $x \neq \inf(E)$. $\boxtimes$

Thm (IVP, top. version) Let X, Y top. spaces, $f: X \to Y$ continuous, $\mathrm{Dom}(f) = X$. Then

$$X \text{ connected} \implies f(X) \text{ connected}$$

Pf WLOG $f(X) = Y$

Let $E \subseteq Y$ be open, E^c open, $E \neq \emptyset$, $E^c \neq \emptyset$.

Then $f^{-1}(E)$, $f^{-1}(E^c)$ open, $\neq \emptyset$, $X = f^{-1}(E) \cup f^{-1}(E^c)$

So $f(X)$ NOT connected $\implies X$ NOT connected $\boxtimes$