Uniform (and Cauchy) continuity

Thm Let $f: X \to Y$ be s.t. X is compact. Then

f continuous $\Rightarrow f$ uniformly continuous

Pf (#1) (sequences) X = seq. compact, f NOT uniformly continuous.
Then $\exists\, \varepsilon > 0 \;\; \exists \{x_n\}, \{y_n\} \in X^{\mathbb{N}}: \; \rho_X(x_n, y_n) \to 0 \wedge \rho_Y(f(x_n), f(y_n)) \geq \varepsilon$

X seq. compact $\Rightarrow \exists x \in X \; \exists \{n_k\}_{k \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}: \; n_k \to \infty \wedge x_{n_k} \to x$.

Then $y_{n_k} \to x$. Then f cannot be continuous at x.

Pf #2 Next board

Pf #2: Let X be compact, f continuous.

Fix $\varepsilon > 0$. For $x \in X$ define

$$\Delta_x := \{\delta \in (0,\infty) : f(B_X(x, 2\delta)) \subseteq B_Y(f(x), \varepsilon/2)\}$$

Continuity of $f \Rightarrow \forall x \in X : \Delta_x \neq \emptyset$.

So $X = \bigcup_{x \in X} \bigcup_{\delta \in \Delta_x} B_X(x, \delta)$

This is an open cover so $\exists$ finite subcover

which means:

$$\exists n \in \mathbb{N}\ \exists z_0 \ldots z_n \in X\ \exists \delta_0 \ldots \delta_n \in (0,\infty) :$$

$$X = \bigcup_{i=0}^{n} B_X(z_i, \delta_i) \wedge \forall i = 0 \ldots n : \delta_i \in \Delta_{z_i}$$

Now set $\delta := \min\{\delta_0, \ldots, \delta_n\} > 0$

Now given $x, y \in X$, set

$$i := \min\{j = 0 \ldots n : x \in B_X(z_i, \delta_i)\}$$

and note $\rho_X(x,y) < \delta$ implies:

$$\rho(y, z_i) \leq \rho(y, x) + \rho(x, z_i) < \delta + \delta_i \leq 2\delta_i$$

So $x, y \in B_X(z_i, 2\delta_i)$

Def of Δ_{z_i} gives

$$f(x), f(y) \in f(B_X(z_i, 2\delta_i)) \subseteq B_Y(f(z_i), \varepsilon/2)$$

So $\rho_Y(f(x), f(y)) \leq \rho_Y(f(x), f(z_i)) + \rho(f(y), f(z_i))$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad \boxtimes$$

Thm (AC) Let X, Y

$f : A \to Y$ with Dom(

(1) A is dense

(2) Y is comple

(3) f is uniform

Then $\exists \bar{f} : X \to Y$

$\mathrm{Dom}(\bar{f}) = X \wedge$

$\wedge\ \bar{f}$ continu

Moreover, the extensi

if g is another funct

$\forall x \in X : g(x) = f(x)$.

f continuous.

$(B_X(x,2\delta)) \subseteq B_Y(f(x),\varepsilon/2)\}$

$\Delta_x \neq \emptyset$.

(x,δ)

finite subcover

$\delta_n \in (0,\infty)$:

$\wedge\ \forall i=0,\dots,n:\ \delta_i \in \Delta_{z_i}$

Now set $\delta := \min\{\delta_0,\dots,\delta_n\} > 0$

Now given $x,y \in X$, set

$i := \min\{j=0,\dots,n:\ x \in B_X(z_i,\delta_i)\}$

and note $\rho_X(x,y) < \delta$ implies:

$\rho(y,z_i) \le \rho(y,x) + \rho(x,z_i) < \delta + \delta_i \le 2\delta_i$

So $x,y \in B_X(z_i,2\delta_i)$

Def of Δ_{z_i} gives

$f(x),f(y) \in f(B_X(z_i,2\delta_i)) \subseteq B_Y(f(z_i),\varepsilon/2)$

So $\rho_Y(f(x),f(y)) \le \rho_Y(f(x),f(z_i)) + \rho(f(y),f(z_i))$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$. $\boxtimes$

Thm (AC) Let X, Y = metric spaces, $A \subseteq X$,
$f : A \to Y$ with $\mathrm{Dom}(f) = A$ obey:

(1) A is dense $(\bar{A} = X)$

(2) Y is complete

(3) f is uniformly (or just Cauchy) continuous (on A)

Then $\exists\, \bar{f} : X \to Y$ with

$\mathrm{Dom}(\bar{f}) = X \ \wedge\ \forall x \in A: \bar{f}(x) = f(x)$

$\wedge\ \bar{f}$ continuous $\quad (*)$

Moreover, the extension is unique, meaning if g is another function satisfying $(*)$ then $\forall x \in X: g(x) = \bar{f}(x)$.

Pf Assume f Cauchy continuous.

Let $G := \{(x, f(x)) : x \in A\} \subseteq X \times Y$

Endow $X \times Y$ with $\tilde{\rho}\left((x,y),(\tilde{x},\tilde{y})\right) := \rho_X(x,\tilde{x}) + \rho_Y(y,\tilde{y})$

Let $\overline{G}$ = closure of G in topology on $X \times Y$.

<u>claim 1 : $\overline{G}$ is a graph of a function</u>

Pf Let $(x,y), (x,\tilde{y}) \in \overline{G}$.

Then $\exists \{x_n\}_{n\in\mathbb{N}}, \{\tilde{x}_n\}_{n\in\mathbb{N}} \in A^{\mathbb{N}}$ s.t.

$$(x_n, f(x_n)) \to (x,y) \wedge (\tilde{x}_n, f(\tilde{x}_n)) \to (x,y)$$

Then $x_n \to x$, $\tilde{x}_n \to x \Rightarrow \rho_Y(f(x_n), f(\tilde{x}_n)) \longrightarrow 0$.

$\Rightarrow \rho_Y(y,\tilde{y}) = 0 \Rightarrow y = \tilde{y}$.

So $\overline{G} = \{ (x, \overline{f}(x)) : x \in \mathrm{Dom}(\overline{f}) \}$ for some $\overline{f} : X \to Y$

Since $G \subseteq \overline{G}$, we get $\forall x \in A : \overline{f}(x) = f(x)$.

Claim 2 $\mathrm{Dom}(\overline{f}) = X$

Pf Let $x \in X$. Then A dense $\Rightarrow \exists \{x_n\}_{n \in \mathbb{N}} \in A^{\mathbb{N}} : x_n \to x$.

Then $\{x_n\}_{n \in \mathbb{N}}$ Cauchy $\Rightarrow \{f(x_n)\}_{n \in \mathbb{N}}$ Cauchy.

Y complete $\Rightarrow \exists y \in Y : f(x_n) \to y$. So $(x, y) \in \overline{G}$

Claim 3 $\overline{f}$ is continuous

Pf Let $\{x_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}}$, $x \in X$ be s.t. $x_n \to x$. G dense in $\overline{G} \Rightarrow$

$\exists \{\tilde{x}_n\}_{n \in \mathbb{N}} \in A^{\mathbb{N}}$ $\forall n \in \mathbb{N} : \rho_X(x_n, \tilde{x}_n) < 2^{-n} \wedge \rho_Y(f(\tilde{x}_n), \overline{f}(x_n)) < 2^{-n}$

Construction of $\overline{f} \wedge \tilde{x}_n \to x \Rightarrow f(\tilde{x}_n) \to \overline{f}(x)$. So we get

$\overline{f}(x_n) \to f(x)$ as well. So $\overline{f}$ is continuous at x. $\boxtimes$

Lemma Pick $a > 0$ (real), let $f : \mathbb{Q} \to \mathbb{R}$ be defined $f(p/q) = \sqrt[q]{a^p}$. Then f is Cauchy continuous an so extends uniquely to a continuous function $\bar{f} : \mathbb{R} \to \mathbb{R}$ s.t. $\forall x \in \mathbb{Q} : \bar{f}(x) = a^x$.

Pf: Note: $\forall x, y \in \mathbb{Q}$:

$$f(x) - f(y) = a^x - a^y = a^y(a^{x-y} - 1)$$

claim $\forall \delta \in (0,1)\ \exists\, n(\delta) \in \mathbb{N}$:

$$1 - \delta < a^{\frac{1}{n(\delta)+1}} < 1 + \delta$$

Pf equivalent to $(1-\delta)^{n(\delta)+1} < a < (1+\delta)^{n(\delta)+1}$

Take $\{x_n\}_{n \in \mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$ Cauchy. Then $M := \sup\{|x_n| : n \in \mathbb{N}\} < \infty$.

Assuming $a > 1$:

$$|f(x_n) - f(x_m)| \leq a^M |a^{x_n - x_m} - 1|$$

So $|x_n - x_m| < \frac{1}{n(\delta)+1} \Rightarrow |a^{x_n - x_m} - 1| < \delta$

Given $\varepsilon > 0$ choose $\delta := a^{-M}\varepsilon$ to get

$$|x_n - x_m| < \frac{1}{n(\delta)+1} \Rightarrow |f(x_n) - f(x_m)| < \varepsilon$$

$\bar{\mathbb{Q}} = \mathbb{R}$, $\mathbb{R}$ complete $\Rightarrow$ f extends continuously. $\boxtimes$

Thm (AC) Le…

$f : A \to Y$ w…

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(2) Y is …

(3) f is …

Then $\exists\, \bar{f} : X$…

$\mathrm{Dom}(\bar{f}) = X$

$\wedge$ $\bar{f}$ co…

Moreover, the …

if g is another…

$\forall x \in X : g(x) =$…