Uniform convergence

Question: Given a two dimensional array $\{a_{m,n}\}_{m,n\in\mathbb{N}}$ of reals, when can we guarantee that the equality

$$\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n} = \lim_{n\to\infty}\lim_{m\to\infty} a_{m,n}$$

holds, assuming the stated limits exist?

Example: Taking $A := \{(k,\ell)\in\mathbb{N}\times\mathbb{N}\colon k\leqslant \ell\}$, note that

$$\lim_{m\to\infty} 1_A(m,n) = 0 \quad\text{and}\quad \lim_{n\to\infty} 1_A(m,n) = 1$$

so exchange of limits is not automatic.

Lemma (Uniformity allows for exchange of limits)

Let $\{a_{m,n}\}_{m\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ *obey*

$$\forall m \in \mathbb{N}\colon \;\; b_m := \lim_{n\to\infty} a_{m,n} \text{ exists}$$

and

$$\forall n \in \mathbb{N}\colon \;\; c_n := \lim_{m\to\infty} a_{m,n} \text{ exists}$$

Assuming, in addition, that the first limit is uniform in the sense

$$\lim_{n\to\infty} \sup_{m\in\mathbb{N}} |b_m - a_{m,n}| = 0$$

then

$$\lim_{m\to\infty} b_m \text{ exists} \,\wedge\, \lim_{n\to\infty} c_n \text{ exists} \,\wedge\, \lim_{m\to\infty} b_m = \lim_{n\to\infty} c_n$$

Two modes of convergence

Definition (Pointwise and uniform convergence)

Given sets $A \subseteq B$, a metric space (X, ρ), a function $f \colon B \to X$ and a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ from B to X (with all having domain B), we say:

- $f_n \to f$ **pointwise on** A if

$$\forall x \in A\colon \ \lim_{n\to\infty} \rho\big(f_n(x), f(x)\big) = 0$$

- $f_n \to f$ **uniformly on** A if

$$\lim_{n\to\infty} \sup_{x\in A} \rho\big(f_n(x), f(x)\big) = 0$$

If $A = B$ then "on A" is usually dropped.

Note: Uniform convergence is metric convergence w.r.t. the **supremum metric**

$$\rho_\infty(f, g) := \sup_{x\in A} \rho\big(f(x), g(x)\big)$$

Lemma

For any functions $\{f_n\}_{n\in\mathbb{N}}$ *and* f*,*

$$f_n \to f \text{ uniformly } \Rightarrow f_n \to f \text{ pointwise}$$

Proof: Uniform convergence implies that, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geqslant n_0 \colon \sup_{x\in A} \rho\big(f_n(x), f(x)\big) < \epsilon$$

But then for any $z \in A$,

$$\forall n \geqslant n_0 \colon \rho\big(f_n(z), f(z)\big) < \epsilon$$

proving pointwise convergence. $\square$

$$f_n \to f \text{ pointwise } :=\ \forall \epsilon > 0\, \forall x \in X\, \exists n_0 \in \mathbb{N}\, \forall n \geqslant n_0 :\ \rho\big(f_n(x), f(x)\big) < \epsilon$$

$$f_n \to f \text{ uniformly } :=\ \forall \epsilon > 0\, \exists n_0 \in \mathbb{N}\, \forall x \in X\, \forall n \geqslant n_0 :\ \rho\big(f_n(x), f(x)\big) < \epsilon$$

Motivating example: For each $x \geqslant 0$,

$$f_n(x) := \frac{nx}{1+nx}$$

converges to $1_{(0,\infty)}(x)$. Yet limit not uniform because $\sup_{x>0} |f_n(x) - 1| = 1$.

What's the obstruction?

Each f_n is continuous yet their pointwise limit is NOT!

Theorem

Given metric spaces X and Y, let $\{f_n\}_{n\in\mathbb{N}}$ and f be functions $X \to Y$ (with domain X) such that $f_n \to f$ uniformly. Then

$$\forall x_0 \in X\colon \ \big(\forall n \in \mathbb{N} : f_n \text{ continuous at } x_0\big) \ \Rightarrow \ f \text{ continuous at } x_0$$

In particular,

$$\big(\forall n \in \mathbb{N} : f_n \text{ continuous}\big) \ \wedge \ f_n \to f \text{ uniformly} \ \Rightarrow \ f \text{ continuous}$$

Let $\epsilon > 0$. Uniform convergence implies existence of an $n \in \mathbb{N}$ such that

$$\forall x \in X\colon\; \rho_Y\big(f_n(x), f(x)\big) < \epsilon$$

Continuity of f_n at x_0 in turn gives a $\delta > 0$ such that

$$\forall x \in X\colon\; \rho_X(x, x_0) < \delta \;\Rightarrow\; \rho_Y\big(f_n(x), f_n(x_0)\big) < \epsilon$$

By the triangle inequality, once $\rho(x, x_0) < \delta$, we get

$$\begin{aligned} \rho_Y\big(f(x), f(x_0)\big) &\leqslant \rho_Y\big(f(x), f_n(x)\big) \\ &\qquad + \rho_Y\big(f(x_0), f_n(x_0)\big) + \rho_Y\big(f_n(x), f_n(x_0)\big) < 3\epsilon \end{aligned}$$

proving continuity of f at x_0. $\square$

Definition

Let A be a set and (X, ρ) a metric space. A sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ from A to X is said to be **uniformly Cauchy** if

$$\forall \epsilon > 0\, \exists n_0 \in \mathbb{N}\, \forall m, n \geqslant n_0 \forall x \in A\colon\ \rho\big(f_n(x), f_m(x)\big) < \epsilon$$

Note: Cauchy w.r.t. the supremum metric $\rho_\infty(f, g) := \sup_{x \in A} \rho(f(x), g(x))$.

Lemma

Let $\{f_n\}_{n\in\mathbb{N}}$ and f be functions from a set A to a metric space X. Then

$$f_n \to f \text{ uniformly} \;\Rightarrow\; \{f_n\}_{n\in\mathbb{N}} \text{ uniformly Cauchy}$$

On the other hand, if (X,ρ) is complete, then

$$\{f_n\}_{n\in\mathbb{N}} \text{ uniformly Cauchy} \;\Rightarrow\; \exists f \in X^A\colon\, f_n \to f \text{ uniformly}$$

Proof: First part follows from

$$\sup_{x\in X} \rho\big(f_n(x), f_m(x)\big) \leqslant \sup_{x\in X} \rho\big(f_n(x), f(x)\big) + \sup_{x\in X} \rho\big(f(x), f_m(x)\big)$$

For the second part note that $\{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy for each x. Completeness lets us define $f(x) := \lim_{n\to\infty} f_n(x)$. Uniformity checked via

$$\rho\big(f(x), f_n(x)\big) = \lim_{m\to\infty} \rho\big(f_m(x), f_n(x)\big) \qquad \square$$

Lemma (Weierstrass M-test)

Let X be a metric space and $\{f_n\}_{n\to\infty}$ functions $X \to \mathbb{R}$ such that, for a sequence $\{M_n\}_{n\in\mathbb{N}} \in [0,\infty)^{\mathbb{N}}$,

$$\forall x \in X\, \forall n \in \mathbb{N}\colon\ \big|f_n(x)\big| \leqslant M_n$$

and

$$\sum_{n=0}^{\infty} M_n < \infty$$

Then there exists a function $f\colon X \to \mathbb{R}$ such that

$$\sum_{k=0}^{n} f_k \to f \ \text{ uniformly}$$

Proof: For $n \leqslant m$ we have

$$\left|\sum_{k=0}^{n} f_k(x) - \sum_{k=0}^{m} f_k(x)\right| \leqslant \sum_{k=n+1}^{m} \big|f_k(x)\big| \leqslant \sum_{k=n+1}^{\infty} M_k$$

so $\{\sum_{k=0}^{n} f_k\}_{n\in\mathbb{N}}$ is uniformly Cauchy. Now use that $\mathbb{R}$ is complete. $\square$

Lemma

Let $\{a_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be such that (with conventions $+\infty^{-1} := 0$ and $0^{-1} := +\infty$)

$$R := \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1} > 0$$

Let $x_0 \in \mathbb{R}$. Then for all $x \in (x_0 - R, x_0 + R)$,

$$f(x) := \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges absolutely and defines a continuous function $f: (x_0 - R, x_0 + R) \to \mathbb{R}$.

Note: R is called the **radius of convergence**

Let $r \in (0, R)$ and let $\epsilon \in (0, R - r)$. As $(R - \epsilon)^{-1} > R^{-1}$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geqslant n_0 \colon \quad |a_n| \leqslant \frac{1}{(R - \epsilon)^n}$$

Setting $A := \max(\{|a_k|(R - \epsilon)^k \colon k = 0, \ldots, n_0\} \cup \{1\})$, we then have

$$\forall n \in \mathbb{N} \colon \quad |a_n| \leqslant \frac{A}{(R - \epsilon)^n}$$

Denoting $f_n(x) := a_n(x - x_0)^n$, we then have

$$M_n := \sup_{x \in [x_0 - r, x_0 + r]} |f_n(x)| \leqslant |a_n| r^n \leqslant A \left(\frac{r}{R - \epsilon} \right)^n$$

As $r < R - \epsilon$, this is summable on n and so the Weierstrass M-test applies. Hence, for all $r \in (0, R)$,

$$f(x) := \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges uniformly on $[x_0 - r, x_0 + r]$ and is continuous there. $\square$

Theorem

Let $a < b$ be reals and $\{f_n\}_{n\in\mathbb{N}}$ and f functions $[a,b] \to \mathbb{R}$ such that

$$\Big(\forall n \in \mathbb{N} : f_n \text{ Riemann integrable}\Big) \wedge f_n \to f \text{ uniformly}$$

Then

$$f \text{ Riemann integrable} \wedge \lim_{n\to\infty} \int_a^b f_n(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x$$

Note: Osgood's Bounded Convergence Theorem does the same assuming only that $f_n \to f$ pointwise with f Riemann integrable.

First note that

$$\left|\int_a^b f_m(x)\mathrm{d}x - \int_a^b f_n(x)\mathrm{d}x\right| = \left|\int_a^b (f_n - f_m)(x)\mathrm{d}x\right| \leqslant (b-a) \sup_{x\in[a,b]} \left|f_n(x) - f_m(x)\right|$$

and since $\{f_n\}_{n\in\mathbb{N}}$ is uniformly Cauchy, $\{\int_a^b f_n(x)\mathrm{d}x\}_{n\in\mathbb{N}}$ is Cauchy and

$$L := \lim_{n\to\infty} \int_a^b f_n(x)\mathrm{d}x$$

exists. Fix $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geqslant n_0 \colon \quad \left|\int_a^b f_n(x)\mathrm{d}x - L\right| < \epsilon$$

Next, uniform convergence $f_n \to f$ implies existence of $n \geqslant n_0$ such that

$$\sup_{x\in[a,b]} \left|f_n(x) - f(x)\right| < \frac{\epsilon}{b-a}$$

which then implies …

... that for all marked partitions Π of $[a, b]$

$$\left|R(f,\Pi) - R(f_n,\Pi)\right| \leqslant (b-a) \sup_{x\in[a,b]} \left|f_n(x) - f(x)\right| < \epsilon$$

The Riemann integrability of f_n in turn gives us $\delta > 0$ such that for all marked partitions Π with $\|\Pi\| < \delta$,

$$\left|R(f_n,\Pi) - \int_a^b f_n(x)\mathrm{d}x\right| < \epsilon$$

The triangle inequality then yields

$$\begin{aligned}\left|R(f,\Pi) - L\right| &\leqslant \left|R(f,\Pi) - R(f_n,\Pi)\right| \\ &\quad + \left|R(f_n,\Pi) - \int_a^b f_n(x)\mathrm{d}x\right| + \left|\int_a^b f_n(x)\mathrm{d}x - L\right| < 3\epsilon\end{aligned}$$

As this holds for all $\epsilon > 0$, we get the claim. $\square$

Theorem

Given reals $a < b$, let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of differentiable functions $(a, b) \to \mathbb{R}$ such that

$$\exists x_0 \in (a, b) : \ \lim_{n\to\infty} f_n(x_0) \text{ exists}$$

and

$$\{f_n'\}_{n\in\mathbb{N}} \text{ is uniformly Cauchy}$$

Then there exists a differentiable function $f : (a, b) \to \mathbb{R}$ such that

$$f_n \to f \text{ uniformly } \ \wedge \ f_n' \to f' \text{ uniformly}$$

In particular, the derivative can be exchanged with the limit $n \to \infty$.

Define $\phi_n\colon (a,b) \to \mathbb{R}$ by

$$\phi_n(x) := \begin{cases} \frac{f_n(x)-f_n(x_0)}{x-x_0} & \text{if } x \neq x_0 \\ f_n'(x_0) & \text{if } x = x_0 \end{cases}$$

Then ϕ_n is continuous on (a,b). Moreover, Lagrange's MVT implies

$$\forall x \in (a,b)\ \exists \xi \in (a,b)\colon\ \ \phi_n(x) - \phi_m(x) = f_n'(\xi) - f_m'(\xi)$$

and so $\{f_n'\}_{n\in\mathbb{N}}$ being uniformly Cauchy implies $\{\phi_n\}_{n\in\mathbb{N}}$ being uniformly Cauchy. In particular,

$$\forall x \in (a,b)\colon\ \ \phi(x) := \lim_{n\to\infty} \phi_n(x) \text{ exists}$$

and, since $\phi_n \to \phi$ uniformly, ϕ is continuous. Next ...

…define

$$f(x) := \lim_{n\to\infty} f_n(x_0) + \phi(x)(x - x_0)$$

and note that, for $x \neq x_0$,

$$f_n(x) = f_n(x_0) + \phi_n(x)(x - x_0)$$

implies that $f_n \to f$ uniformly. This means that we can repeat the above arguments with x_0 replaced by any x. It thus suffices to show that $f'(x_0)$ exists and $f_n'(x_0) \to f'(x_0)$. Here we note that, for $x \neq x_0$,

$$\begin{aligned}\frac{f(x) - f(x_0)}{x - x_0} - \lim_{n\to\infty} f_n'(x_0) &= \lim_{n\to\infty} \left[\frac{f_n(x) - f_n(x_0)}{x - x_0} - f_n'(x_0) \right] \\ &= \lim_{n\to\infty} \left[\phi_n(x) - \phi_n(x_0) \right] = \phi(x) - \phi(x_0)\end{aligned}$$

By continuity of ϕ, this tends to zero as $x \to x_0$. $\square$

Corollary

The power series

$$f(x) := \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is infinitely differentiable on $(x_0 - R, x_0 + R)$ *once*

$$R := \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1} > 0$$

Proof: Set $f_n(x) := \sum_{k=0}^{n} a_k (x - x_0)^k$. Then

$$f_n'(x) = \sum_{k=1}^{n} k a_k (x - x_0)^{k-1}$$

Now use that for $x \in [x_0 - r, x_0 + r]$ we have $k|a_k||x - x_0|^{k-1} \leqslant Ak(\frac{r}{R-\epsilon})^k r^{-1}$ which is still summable on k once $0 < r < R - \epsilon$.

Lemma (Exponential, sine and cosine)

The real-valued functions exp, sin *and* cos *are well-defined for all* $x \in \mathbb{R}$ *by*

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

where the series converge absolutely everywhere and uniformly on any compact subinterval of $\mathbb{R}$. *Moreover, these functions are infinitely differentiable on* $\mathbb{R}$ *with*

$$\exp'(x) = \exp(x) \;\wedge\; \sin'(x) = \cos(x) \;\wedge\; \cos'(x) = -\sin(x)$$

at each $x \in \mathbb{R}$.