# Stieltjes integral

**Def** Given $f, g : [a,b] \to \mathbb{R}$ and marked partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$

set $$S(f, dg, \Pi) := \sum_{i=1}^n f(t_i^*)\big(g(t_i) - g(t_{i-1})\big)$$

We say that $f$ is Riemann-Stieltjes integrable w.r.t $g$ on $[a,b]$ if

$$\int_a^b f\,dg := \lim_{\|\Pi\| \to 0} S(f, dg, \Pi)$$

We call ↗ the (Riemann)-Stieltjes integral.

## Motivation:

- physics: work of force along a path

$$\int_0^T F(t)\cdot dx(t)$$

- economics: portfolio price $g(t)$ at time $t$
  position in it $f(t)$ — " — $t$

  then total change of value over the interval $[T_1, T_2]$ equals $\int_{T_1}^{T} f\,dg$

- probability: $g(t) = P(X \le t)$

$$E(f(X)) := \int_a^b f\,dg \quad \left(\begin{array}{l} X \text{ takes} \\ \text{values in } [a,b] \end{array}\right)$$

## Properties of RS-integral.

Notation:

$$RS(g,[a,b]) := \left\{ f \in \mathbb{R}^{[a,b]} : \int_a^b f\,dg \text{ exists} \right\}$$

**Lemma** Let $f_1, f_2, g : [a,b] \to \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$. Then

$$f_1, f_2 \in RS(g,[a,b]) \Rightarrow \alpha f_1 + \beta f_2 \in RS(g,[a,b])$$

$$\wedge \int_a^b (\alpha f_1 + \beta f_2)\,dg = \alpha \int_a^b f_1\,dg + \beta \int_a^b f_2\,dg$$

**Lemma** Let $f, g_1, g_2 : [a,b]$, $\alpha, \beta \in \mathbb{R}$. Then

$$f \in RS(g_1,[a,b]) \cap RS(g_2,[a,b]) \Rightarrow f \in RS(\alpha g_1 + \beta g_2, [a,b])$$

$$\wedge \int_a^b f\,d(\alpha g_1 + \beta g_2) = \alpha \int_a^b f\,dg_1 + \beta \int_a^b f\,dg_2$$

**Lemma** (Additivity)

$f \in RS(g,[a,b]) \Rightarrow$

$\wedge \int_a^b f\,dg = \int_a^c$

Pf: Force $c$ as a p
and apply Cauch

**Lemma** (Necessary cond

Let $f, g : [a,b] \to \mathbb{R}$ be

Then $\forall \varepsilon > 0\ \exists \delta > 0\ \forall \Pi$

$\|\Pi\| < \delta \Rightarrow$

...force along a path

...$\cdot dx(t)$

...price $g(t)$ at time $t$

... in it $f(t)$ —"— $t$

... of value over

$[T_1,T]$ equals $\int_{T_1}^{T} f\,dg$

...$(t) = P(X \le t)$

$= \int_a^b f\,dg$ ($X$ takes values in $[a,b]$)

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**Lemma** (Additivity) For $a<c<b$, $f,g : [a,b] \to \mathbb{R}$:

$f \in RS(g,[a,b]) \Rightarrow f \in RS(g,[a,c]) \cap RS(g,[c,b])$

$\wedge \int_a^b f\,dg = \int_a^c f\,dg + \int_c^b f\,dg$

Pf: Force $c$ as a partition point and apply Cauchy criterion in each half.

**Lemma** (Necessary cond for RS integrability)

Let $f,g : [a,b] \to \mathbb{R}$ be s.t. $f \in RS(g,[a,b])$.

Then $\forall \varepsilon > 0\ \exists \delta > 0\ \forall \Pi =$ unmarked partition:

$$\|\Pi\| < \delta \Rightarrow \sum_{i=1}^{n} osc(f,[t_{i-1},t_i])\,|g(t_i) - g(t_{i-1})| < \varepsilon$$

Pf Let $\delta$ relate to $\varepsilon$ as in def of RS-integrability.
Then given unmarked partn. $\Pi$, let $\Pi^*, \Pi^{**}$
marked partitions based on $\Pi$ s.t.

$$\forall i = 1 \dots n: \quad |f(t_i^*) - f(t_i^{**})| \geq \frac{1}{2} \operatorname{osc}(f, [t_{i-1}, t_i])$$

$$\wedge \quad f(t_i^*) - f(t_i^{**}) \text{ has same sign as } g(t_i) - g(t_{i-1}).$$

Then

$$2\varepsilon \geq S(f, dg, \Pi^*) - S(f, dg, \Pi^{**})$$

$$= \sum_{i=1}^{n} |f(t_i^*) - f(t_i^{**})| \, |g(t_i) - g(t_{i-1})|$$

$$\geq \frac{1}{2} \sum_{i=1}^{n} \operatorname{osc}(f, [t_{i-1}, t_i]) \, |g(t_i) - g(t_{i-1})| \qquad \boxtimes$$

**Corollary:** If $f \in RS(g, [a,b])$ then $f$ and $g$ do NOT have a common discontinuity point.

Pf NOTES.

This also prevents additivity to be iff.

$$\boxed{dg = g'(x)\,dx}$$

**Lemma** (Reduction to Riemann). Let $f, g : [a,b] \to \mathbb{R}$ be s.t.

1) $f$ RI on $[a,b]$
2) $g$ cont on $[a,b]$, diff on $(a,b)$, $g'$ is RI on $[a,b]$

Then $f \in RS(g, [a,b]) \wedge \int_a^b f\,dg = \int_a^b f(x)\,g'(x)\,dx$ $\boxed{\text{Pf: HW7}}$

**Lemma** (Integration by parts)

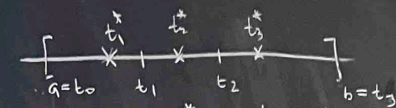
Let $f, g : [a,b] \to \mathbb{R}$. Then

$$f \in RS(g, [a,b]) \iff g \in RS(f, [a,b])$$

and if both TRUE, then

$$\int_a^b f\,dg = f(b)g(b) - f(a)g(a) - \int_a^b g\,df$$

**Pf**: Let $\Pi = (\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$




Set $t_0^* := a$, $t_{n+1}^* := b$ and note $\Pi' = (\{t_i^*\}_{i=0}^{n+1}, \{t_{i-1}\}_{i=1}^n)$ is also a marked partition. Now:

$$f(a)g(a) + S(f, dg, \Pi)$$
$$= f(a)g(a) + \sum_{i=1}^n f(t_i^*)[g(t_i) - g(t_{i-1})]$$
$$= \sum_{i=0}^n f(t_i^*)g(t_i) - \sum_{i=0}^n f(t_{i+1}^*)g(t_i) + f(b)g(b)$$

$$= \sum_{i=1}^{n+1} g($$

$$= S(g, df$$

Note $\|\Pi'\| \leq$

So $\|\Pi\| \to 0$

and $\lim_{\|\Pi\| \to 0} S(f$

Now read the

...tion by parts)

$\to \mathbb{R}$. Then

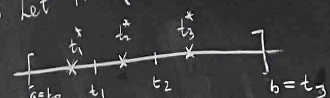
$b]) \Leftrightarrow g \in RS(f, [a,b])$

...RE, then

$)g(b) - f(a)g(a)$

$- \int_a^b g\,df$

Pf: Let $\Pi = (\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$

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$$f(a)g(a) + S(f, dg, \Pi)$$
$$= f(a)g(a) + \sum_{i=1}^n f(t_i^*)[g(t_i) - g(t_{i-1})]$$
$$= \sum_{i=0}^n f(t_i^*)g(t_i) - \sum_{i=0}^n f(t_{i+1}^*)g(t_i) + f(b)g(b)$$
$$= -\sum_{i=1}^{n+1} g(t_{i-1})[f(t_i^*) - f(t_{i-1}^*)] + f(b)g(b)$$
$$= S(g, df, \Pi') + f(b)g(b)$$

Note $\|\Pi'\| \le 2\|\Pi\| \wedge \|\Pi\| \le 2\|\Pi'\|$

So $\|\Pi\| \to 0 \Leftrightarrow \|\Pi'\| \to 0$

and $\lim_{\|\Pi\| \to 0} S(f, dg, \Pi)$ exists

$\Leftrightarrow \lim_{\|\Pi'\| \to 0} S(g, df, \Pi')$ exists $\therefore$

Now read the formula in the limit. $\boxtimes$

Lemma (Sub rule) Let $g, h : [a,b] \to \mathbb{R}$ be s.t. $g \in RS(h, [a,b])$. Set

$$G(x) := \int_a^x g \, dh \qquad x \in [a,b].$$

Let $f : [a,b] \to \mathbb{R}$. Then

$$f \in RS(G, [a,b]) \iff f \cdot g \in RS(h, [a,b])$$

and if both TRUE, then

$$\int_a^b f \, dG = \int_a^b f \cdot g \, dh$$

Pf. NOTES

Suff cond. for $f \in RS(g, [a,b])$ is

$f$ continuous $\wedge$ $g \in BV([a,b])$

but it pays off to trade regularity of one function for the other. E.g. suffices

$f$ $\alpha$-Hölder $\wedge$ $g$ $\beta$-Hölder for $\alpha, \beta \in (0,1]$ s.t. $\alpha + \beta > 1$

"Young integral"