What lies beyond Riemann integral

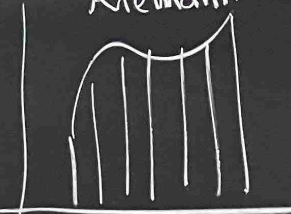
— Lebesgue integration theory

key idea: partition range instead of domain

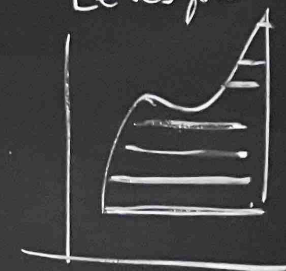
$$\sum_{k\in\mathbb{N}} \frac{k}{N}\,\text{length}\left(f^{-1}\left(\left[\frac{k}{N},\frac{k+1}{N}\right)\right)\right)$$

replaces Riemann's sum.

Riemann



Lebesgue



Issue: Need robust theory of length.

$$\lambda^*(A) := \inf\left\{\sum_{i=0}^{\infty}(b_i-a_i) : A \subseteq \bigcup_{i\in\mathbb{N}}(a_i,b_i)\right\}$$

Henstock - Kurzweil integral

idea: Force partition to be finer around points where integrand behaves badly.

Def Let $a<b$, $\gamma : [a,b] \longrightarrow (0,\infty)$ (called gauge) be given. A marked partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$ is said to be γ-fine if

$$\forall i=1,\dots,n: \quad [t_{i-1}, t_i] \subseteq [t_i^* - \gamma(t_i^*), t_i^* + \gamma(t_i^*)]$$

Q: Are there any γ-fine partitions at all?

Thm (Cousin 1895) Let $\mathcal{I}$ be a set of non-degenerate closed subintervals of $[a,b]$ s.t.

$$\forall x\in[a,b]\ \exists \delta>0\ \forall [c,d]\subseteq[a,b]: \ 0<d-c<\delta \wedge x\in[c,d] \Rightarrow [c,d]\in\mathcal{I}.$$

Then $\exists \Pi = (\{t_i\}_{i=1}^n)$ partition of $[a,b]$ s.t. $\forall i=1\dots n: [t_{i-1}, t_i] \in \mathcal{I}$.

Pf. HW7.

Note $\mathcal{I} = \bigcup_{t\in[a,b]} \{[c,$ …

has Cousin property …

exist for all $\gamma: [a,b$ …

Def Let $f: [a,b] \to \mathbb{R}$ …

Henstock-Kurzweil …

if

$\exists L\in\mathbb{R}\ \forall \varepsilon>0\ \exists \gamma\in(0,\infty)$ …

Π is γ-fine $\Rightarrow$ …

Henstock-Kurzweil integral

a: Force partition to be finer around pts where integrand behaves badly.

Def Let $a<b$, $\gamma:[a,b]\to(0,\infty)$ (called gauge) be given. A marked partition $\Pi=(\{t_i\}_{i=0}^n,\{t_i^*\}_{i=1}^n)$ is said to be γ-fine if

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Then $\exists\Pi=(\{t_i\}_{i=1}^n)$ partition of $[a,b]$ s.t. $\forall i=1\ldots n:\ [t_{i-1},t_i]\in\mathcal{I}$.

Pf. HW.

Note

$$\mathcal{I}=\bigcup_{t\in[a,b]}\left\{[c,d]:\ \begin{array}{l}a\le c\le t\le d\le b,\\ 0<d-c<\gamma(t)\end{array}\right\}$$

has Cousin property and so γ-fine partitions exist for all $\gamma:[a,b]\to(0,\infty)$.

Def Let $f:[a,b]\to\mathbb{R}$. We say that f is Henstock-Kurzweil integrable (HKI) on $[a,b]$ if

$$\exists L\in\mathbb{R}\ \forall\varepsilon>0\ \exists\gamma\in(0,\infty)^{[a,b]}\ \forall\Pi=\text{marked partition of }[a,b]:$$
$$\Pi \text{ is } \gamma\text{-fine} \Rightarrow |R(f,\Pi)-L|<\varepsilon$$

We call the unique L in above def. the HK-integral, with notation $\int_a^b f(x)\,dx$.

Lemma f RI on $[a,b] \Rightarrow f$ HKI on $[a,b]$

Pf. RI ... $\forall \varepsilon > 0 \ \exists \delta > 0 \ \ldots \ \forall \Pi : \ \|\Pi\| < \delta \Rightarrow \ldots$

Choose $\gamma(t) := \delta/2$. $\boxtimes$

Lemma $1_{\mathbb{Q}}$ is HKI on $[a,b]$, for all $a<b$.

Pf. Enumerate $\mathbb{Q} = \{q_n\}_{n \in \mathbb{N}}$. Define:

$$\gamma(t) := \begin{cases} 1 & t \notin \mathbb{Q} \\ \frac{\varepsilon}{2^n} & t = q_n \end{cases}$$

then $\forall \Pi$ partitions s.t. Π is γ-fine:

$$0 \le R(f, \Pi) = \sum_{i=1}^{n} 1_{\mathbb{Q}}(t_i^*)(t_i - t_{i-1})$$

$$\le \sum_{k \in \mathbb{N}} 2 \frac{\varepsilon}{2^k} = 2\varepsilon$$

So we get that $1_{\mathbb{Q}}$ is HK integrable and $\int_a^b 1_{\mathbb{Q}}(x)\,dx = 0$.

HK integral handles improper integrals automatically:

$$\int_0^1 \frac{dx}{\sqrt{x}}, \quad \int_0^1 \frac{\sin(1/x)}{x}\,dx$$

Thm (Hake's thm) Suppose $f : [a,b] \to \mathbb{R}$ is s.t.

- $\forall c \in (a,b]$: f HKI on $[c,b]$
- $\lim_{c \to a^+} \int_c^b f(x)\,dx$ exists.

Then f HKI on $[a,b]$ $\wedge$ $\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$

Thm (FTC II in HK theory)

Let $F : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b). Choosing $F'(a), F'(b)$ arbitrarily:

F' is HKI on $[a,b]$

$\wedge$ $\int_a^b F'(x)\,dx = F(b) - F(a)$.

Pf Assume F

$\gamma(t) := \frac{1}{2}$

Then for $\Pi =$

$F(b) - F(a) -$

$= \sum_{i=1}^{n} [F$

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$f: [a,b] \to \mathbb{R}$

I on $[a,b]$

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Thm (FTC II in HK theory)

Let $F: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b). Choosing $F'(a), F'(b)$ arbitrarily:

F' is HKI on $[a,b]$

$\wedge \int_a^b F'(x)\,dx = F(b) - F(a).$

Pf Assume $F'(x)$ exists $\forall x \in (a,b)$. For $t \in (a,b)$:

$$\gamma(t) := \frac{1}{2} \sup\left\{ \delta \in (0,1] : \sup_{\substack{x \in [a,b] \\ 0 < |x-t| < \delta}} \left| \frac{F(x) - F(t)}{x - t} - F'(t) \right| < \frac{\varepsilon}{b-a} \right\}$$

Then for $\Pi = \left(\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n \right)$ γ-fine:

$$F(b) - F(a) - R(F', \Pi)$$
$$= \sum_{i=1}^n \left[F(t_i) - F(t_{i-1}) - F'(t_i^*)(t_i - t_{i-1}) \right]$$
$$= \sum_{i=1}^n \left[\left(F(t_i) - F(t_i^*) - F'(t_i^*)(t_i - t_i^*) \right) - \left(F(t_{i-1}) - F(t_i^*) - F'(t_i^*)(t_i^* - t_{i-1}) \right) \right]$$

$$\left| F(b) - F(a) - R(F', \Pi) \right|$$

$$\leq \sum_{i=1}^{n} \left[\frac{\varepsilon}{b-a} (t_i - t_i^*) + \frac{\varepsilon}{b-a} (t_i^* - t_{i-1}) \right]$$

$$= \varepsilon$$

Hence: F' is HKI on $[a,b]$

and $\int_a^b F'(x)\,dx = F(b) - F(a)$ $\boxtimes$