Continuing continuity

<u>recall:</u> ∘ $f : X \to Y$ continuous at x if $\forall_{\varepsilon>0}\ \exists \delta>0 : f(B_X(x,\delta)) \subseteq B_Y(f(x),\varepsilon)$

f continuous $:=$ continuous at x for all $x \in \mathrm{Dom}(f)$.

∘ characterizations: $x_n \to x \Rightarrow f(x_n) \to f(x)$ (sequential)

topological: $(\mathrm{Dom}(f) = X)$ $\forall U \subseteq Y$: open $\Rightarrow f^{-1}(U)$ open

<u>Q:</u> What maps image open sets to open sets? (closed → closed)

<u>Def</u> $f : X \to Y$ is <u>open</u> (an <u>open map</u>) if

$\forall O \subseteq X$: open $\Rightarrow f(O)$ open.

$f : X \to Y$ is <u>closed</u> (a closed map) if

$\forall C \subseteq Y$: closed $\Rightarrow f(C)$ closed

Ex $f(x) = y$ (constant map)
is always closed, not open unless y isolated

Lemma $f : \mathbb{R} \to \mathbb{R}$ strictly increasing, continuous is open.

Pf: Every $O \subseteq \mathbb{R}$ open is a countable union of disjoint open intervals.
A strictly increasing continuous f images every open interval into an open interval.
(using Intermediate Value Thm next time).

Why open good?

Lemma f bijective. Then
f open $\Leftrightarrow$ f^{-1} continuous

Pf Denote $g := f^{-1}$. Then $(f : X \to Y)$
$\forall O \subseteq X$: open $\Rightarrow$ $g^{-1}(O) = f(O)$ open
Applies other way round as well.

Corollary $\forall n \in \mathbb{N} \setminus \{0\}$: f

is continuous.

Pf $h(x) := x^n$ strictly m

is open, so admits

($\mathrm{Ran}(f) = [0, \infty)$ follow

Thm Let $f : X \to Y$ be s

Then X compact $\wedge$ f continuo

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Why open good?

<u>Lemma</u> f bijective. Then

f open $\Leftrightarrow f^{-1}$ continuous

<u>Pf</u> Denote $g := f^{-1}$. Then ($f : X \to Y$)

$\forall O \subseteq X$ open $\Rightarrow g^{-1}(O) = f(O)$ open

Applies other way round as well.

<u>Corollary</u> $\forall n \in \mathbb{N} \setminus \{0\}$: $f(x) = x^{1/n}$ ($\mathrm{Dom}(f) = [0, \infty)$) is continuous.

<u>Pf</u> $h(x) := x^n$ strictly increasing, cont function is open, so admits a continuous inverse. ($\mathrm{Ran}(f) = [0, \infty)$ follows from IVT.)

<u>Thm</u> Let $f : X \to Y$ be s.t. $\mathrm{Dom}(f) = X$.

Then X compact $\wedge$ f continuous $\Rightarrow f(X)$ compact

Pf #1) Let $\{y_n\}_{n\in\mathbb{N}} \in f(X)^{\mathbb{N}}$. Then (by AC) $\exists \{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ s.t. $\forall n\in\mathbb{N}: f(x_n) = y_n$.

X (seq.) compact $\Rightarrow \exists \{n_k\}_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}} \exists x\in X:$

$n_k \to \infty \wedge x_{n_k} \to x$.

f cont $\Rightarrow f(x_{n_k}) \to f(x)$. So $y_{n_k} \to f(x) \in f(X)$.

So $f(X)$ is (seq.) compact.

#2) Let $\{O_\alpha\}_{\alpha\in I}$ is open cover of $f(X)$.

Then $\{f^{-1}(O_\alpha)\}_{\alpha\in I}$ is open cover of X.

f continuous $\Rightarrow f^{-1}(O_\alpha)$ is open $(\forall\alpha\in I)$.

X compact $\Rightarrow \exists F\subseteq I$: finite $\wedge \bigcup_{\alpha\in F} f^{-1}(O_\alpha) \supseteq X$.

Then $\bigcup_{\alpha\in F} O_\alpha \supseteq f\left(\bigcup_{\alpha\in F} f^{-1}(O_\alpha)\right) \supseteq f(X)$

So $f(X)$ is compact.

<u>Corollary</u> Let $f: X \to \mathbb{R}$ ($\mathrm{Dom}(f) = X$), let $K \subseteq X$.
Assume f continuous and K compact.
Then $f(K)$ is bounded and $\exists\, x_0, x_1 \in K$:
$$f(x_0) = \inf\{f(x) : x \in K\} \wedge f(x_1) = \sup\{f(x) : x \in K\}.$$

In short, a continuous function achieves its min / max on every compact set.

<u>Pf:</u> By Thm $f(K)$ is compact in $\mathbb{R}$. Heine-Borel Thm says that $f(K)$ is closed and bounded. Closed $\Rightarrow$ contains all adherent points. Since $\inf_K f$, $\sup_K f$ are adherent, they are achieved in $f(K)$.

Cauchy and uniform continuity

Def A function $f: X \to Y$ is *Cauchy continuous* if

$\forall \{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}: \{x_n\}_{n\in\mathbb{N}}$ Cauchy $\Rightarrow \{f(x_n)\}_{n\in\mathbb{N}}$ Cauchy

Lemma (AC) f Cauchy continuous $\Rightarrow f$ continuous

Pf Let $\{x_n\}_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ be s.t. $x_n \to x$.
Def $z_n := \begin{cases} x_n & n\in 2\mathbb{N} \\ x & n\notin 2\mathbb{N} \end{cases}$. Thm $\{z_n\}$ Cauchy so $\{f(z_n)\}$ Cauchy.
$\Rightarrow f(x_n) \to f(x)$.

Ex $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}$

$f(x) := \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$

Then f is continuous yet $x_n := (-2)^{-n}$ shows f is NOT Cauchy continuous.

Def We say $f: X \to Y$ is *uniformly continuous* if

$\forall \varepsilon>0\ \exists \delta>0\ \forall x,y\in X: \rho_X(x,y)<\delta \Rightarrow \rho_Y(f(x),f(y))<\varepsilon$

$\cancel{(\forall \varepsilon>0\ \forall x\in X\ \exists\delta>0\ \forall y\in X:)}$

Lemma f unifo
$\Rightarrow f$ co
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Thm Let $f: X$ –
Then
f continuo

Pf: Next time

Ex $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$

$$f(x) := \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

Then f is continuous yet $x_n := (-2)^{-n}$ shows f is NOT Cauchy continuous.

Def We say $f : X \to Y$ is uniformly continuous if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in X :\ \rho_X(x,y) < \delta \Rightarrow \rho_Y(f(x), f(y)) < \varepsilon$$

(~~$\forall \varepsilon > 0\ \forall x \in X\ \exists \delta > 0\ \forall y \in X :$~~) ← def. of continuity

Lemma f uniformly continuous

$\Rightarrow f$ continuous

$(\Rightarrow f$ Cauchy continuous$)$

Thm Let $f : X \to Y$ with X compact.

Then

f continuous $\Rightarrow f$ uniformly continuous

Pf: Next time.

$\{x_n\}_{n \in \mathbb{N}}$ Cauchy

ous

uchy

Cauchy.

$f(x)$.