Shortcomings ~~& extension~~ of Riemann integral

recall FTC I & II give:

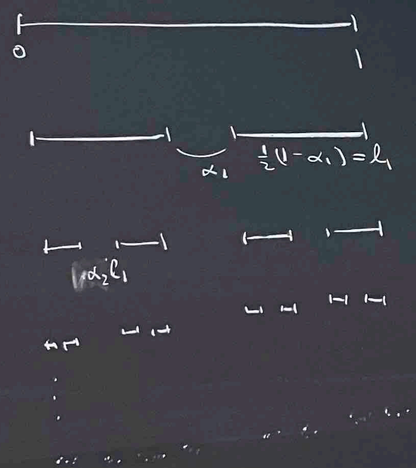
- derivative inverts integral at points of continuity
- integral inverts derivative provided derivative Riemann integrable.

Q: Is the requirement of RI for derivative necessary?

Thm (Volterra's example) There exists $f: \mathbb{R} \to \mathbb{R}$ which continuous, differentiable with f' bounded but NOT RI on $[0,1]$.

Proof modulo some details:

(1) Fat Cantor set $\qquad \{\alpha_n\}_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$



Take $C_0 := [0,1]$.
Assuming C_n is the union of 2^n disjoint closed intervals of length $\ell_n := \left(\prod_{i=1}^{n}(1-\alpha_i)\right)\frac{1}{2^n}$
remove open interval of length $\alpha_{n+1}\ell_n$ from middle of these intervals to get C_{n+1}

$$\ell_{n+1} = \frac{1}{2}(\ell_n - \alpha_{n+1}\ell_n) = \frac{1}{2}(1-\alpha_{n+1})\ell_n$$

Lemma: Let $C := \bigcap_{n\in\mathbb{N}} C_n$. Then C is $\neq \emptyset$, compact, with no isolated points and $\mathbb{R}\setminus C$ dense in $\mathbb{R}$. Moreover,

$$\sum_{i\in\mathbb{N}} \alpha_i < \infty \Rightarrow C \text{ has non-zero length}$$

length of $C_n = 2^n \ell_n = \prod_{i=1}^{n}(1-\alpha_i)$

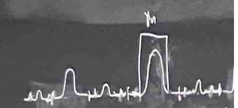
Fact from convergence theory:

$$\lim_{n\to\infty} \prod_{i=1}^{n}(1-\alpha_i) > 0 \Leftrightarrow \forall i: \alpha_i < 1 \ \wedge \ \sum_{i=1}^{\infty} \alpha_i < \infty$$

(2) Inscribing bumps:

$$h(x) = \begin{cases} (1-4x^2)^2 \\ 0 \end{cases}$$

Define f as follows: Pic



explicitly: Let I_n = open int… to get C…

for $x \in (a,b) \in I_n$, set $f(x$…

else set $f(x) = 0$

$\}_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$

Take $C_0 := [0,1]$.
Assuming C_n is the union of 2^n disjoint closed intervals of length $l_n := \left(\prod_{i=1}^{n}(1-\alpha_i)\right)\frac{1}{2^n}$
remove open interval of length $\alpha_{n+1} l_n$ from middle of these intervals to get C_{n+1}

$$l_{n+1} = \frac{1}{2}\left(l_n - \alpha_{n+1} l_n\right) = \frac{1}{2}(1-\alpha_{n+1})\,l_n$$

<u>Lemma</u>: Let $C := \bigcap_{n\in\mathbb{N}} C_n$. Then C is $\neq \emptyset$, compact, with no isolated points and $\mathbb{R}\setminus C$ dense in $\mathbb{R}$. Moreover,

$$\sum_{i\in\mathbb{N}} \alpha_i < \infty \Rightarrow C \text{ has non-zero length}$$

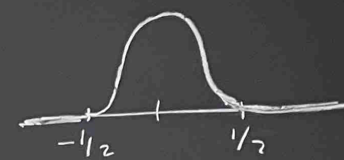
length of $C_n = 2^n l_n = \prod_{i=1}^{n}(1-\alpha_i)$

Fact from convergence theory:

$$\lim_{n\to\infty} \prod_{i=1}^{n}(1-\alpha_i) > 0 \iff \forall i: \alpha_i < 1 \;\wedge\; \sum_{i=1}^{\infty}\alpha_i < \infty$$

(2) <u>Inscribing bumps</u>:

$$h(x) = \begin{cases} (1-4x^2)^2 & |x| \le 1/2 \\ 0 & |x| > 1/2 \end{cases}$$



Define f as follows: Pick $\{\gamma_n\}_{n\in\mathbb{N}} \in (0,1)^{\mathbb{N}}$

explicitly: Let I_n = open intervals removed from C_n to get C_{n+1}.

for $x \in (a,b) \in I_n$, set $f(x) = \gamma_n (b-a)\, h\left(\frac{1}{\gamma_n}\frac{x-\frac{a+b}{2}}{b-a}\right)$
else set $f(x) = 0$

Then for $x \in (a,b) \in I_n$: $f'(x) = h'\left(\dfrac{x - \frac{a+b}{2}}{\gamma_n (b-a)} \right)$

If $x \in C$, $z \notin C$ s.t. $f(z) \neq 0$, then

$\exists n \in \mathbb{N}$ $\exists (a,b) \in I_n$ s.t. $\left| z - \frac{a+b}{2} \right| < \frac{1}{2} \gamma_n (b-a)$

yet $x \in (a,b) \Rightarrow \left| x - \frac{a+b}{2} \right| > \frac{1}{2}(b-a)$

So $|z - x| \geq \frac{1}{2}(b-a) - \frac{1}{2}\gamma_n (b-a) = \frac{1}{2}(1-\gamma_n)(b-a)$

Now $|f(z) - f(x)| = \gamma_n (b-a)\, h\left(\dfrac{z - \frac{a+b}{2}}{\gamma_n (b-a)} \right) \leq \gamma_n (b-a)$

$$\left| \frac{f(z) - f(x)}{z - x} \right| \leq \frac{\gamma_n (b-a)}{\frac{1}{2}(1-\gamma_n)(b-a)} = \frac{2\gamma_n}{1-\gamma_n}$$

So $\lim\limits_{n \to \infty} \gamma_n = 0 \Rightarrow \forall x \in C : f'(x) = 0$

Conclusion f$'$ exists all all points
but is discontinous at all $x \in C$
Since C is NOT of zero length
Lebesgue-Vitali Thm $\Rightarrow$ f' is NOT RI.

Next problem with RI: Integrand must be bounded!
So $\int_0^1 \frac{dx}{\sqrt{x}}$ is NOT meaningful

Improper integral

Def Let $f : (a,b] \to \mathbb{R}$ be s.t.
$\forall c \in (a,b] : \exists \int$ RI on $[c,b]$.

Then
$$\int_a^b f(x)\,dx := \lim_{c \to a^+} \int_c^b f(x)\,dx$$
is the improper integral of f on $[a,b]$, assuming the limit exists.

Similarly we deal with upper limits, or limits $a \to -\infty$, $b \to +\infty$.

Ex: $$\int_0^1 \frac{dx}{\sqrt{x}} := \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{dx}{\sqrt{x}} \overset{\text{FTCII}}{=} \lim_{\varepsilon \to 0^+} 2\sqrt{x}\Big|_\varepsilon^1 = \lim_{\varepsilon \to 0^+} 2 - 2\sqrt{\varepsilon} = 2$$

Ex $$\int_0^y \frac{\sin(1/x)}{x}\,dx := \lim_{\varepsilon \to 0^+} \int_\varepsilon^y \frac{\sin(1/x)}{x}\,dx$$
$$= \lim_{\varepsilon \to 0^+} \int_\varepsilon^y x \frac{d}{dx} \cos(1/x)\,dx$$
$$\overset{\text{IBP}}{=} \lim_{\varepsilon \to 0^+} \left(x\cos(1/x)\Big|_\varepsilon^y - \int_\varepsilon^y \cos(1/x)\,dx \right)$$
$$= y\cos(1/y) - \int_0^y \cos(1/x)\,dx$$

Ex (Fresnel

$\int_0^\infty \sin(x^2$

$x = \sqrt{t}$, $dx = \frac{dt}{2\sqrt{t}}$

dx

[a,b],

or limits

Ex: $\int_0^1 \frac{dx}{\sqrt{x}} := \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{dx}{\sqrt{x}} \overset{\text{FTCII}}{=} \lim_{\varepsilon \to 0^+} 2\sqrt{x}\Big|_\varepsilon^1 = \lim_{\varepsilon \to 0^+} 2 - 2\sqrt{\varepsilon} = 2$

Ex $\int_0^y \frac{\sin(1/x)}{x}\,dx := \lim_{\varepsilon \to 0^+} \int_\varepsilon^y \frac{\sin(1/x)}{x}\,dx$

$= \lim_{\varepsilon \to 0^+} \int_\varepsilon^y x \frac{d}{dx}\cos(1/x)\,dx$

$\overset{\text{IBP}}{=} \lim_{\varepsilon \to 0^+} \left(x\cos(1/x)\Big|_\varepsilon^y - \int_\varepsilon^y \cos(1/x)\,dx \right)$

$= y\cos(1/y) - \int_0^y \cos(1/x)\,dx$

Ex (Fresnel integral)

$$\int_0^\infty \sin(x^2)\,dx = \lim_{M \to \infty} \int_0^M \sin(x^2)\,dx$$

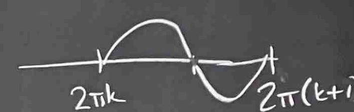
bounded near zero.

$$\underset{\substack{x=\sqrt{t} \\ dx = \frac{dt}{2\sqrt{t}}}}{=} \lim_{M \to \infty} \int_0^{M^2} \frac{\sin(t)}{2\sqrt{t}}\,dt$$

$$= \lim_{n \to \infty} \int_0^{2n\pi} \frac{\sin(t)}{2\sqrt{t}}\,dt$$

$$= \lim_{n \to \infty} \sum_{k=0}^{n-1} \underbrace{\int_{2\pi k}^{2\pi(k+1)} \frac{\sin(t)}{2\sqrt{t}}\,dt}_{\int_0^\pi \frac{\sin(t)}{2}\left[\frac{1}{\sqrt{t+2\pi k}} - \frac{1}{\sqrt{t+2\pi k+\pi}}\right]}$$

$$\frac{1}{\sqrt{t+2\pi k}\sqrt{2\pi k+t+\pi}} \quad \frac{\pi}{\sqrt{2\pi k+t}+\sqrt{2\pi k+t+\pi}}$$



So $\left| \int_{2\pi k}^{2\pi(k+1)} \frac{\sin(t)}{\sqrt{2t}} \right| \leq \frac{\pi}{(2\pi k)^{3/2}}$

So limit exists and $\int_0^\infty \sin(x^2)\,dx$ is defined by it.

Complex analysis:

$$\int_0^\infty \sin(x^2)\,dx = \sqrt{\frac{\pi}{8}}.$$