Fundamental Theorem of Calculus

Def Given $f : [a,b] \to \mathbb{R}$, its antiderivative on $[a,b]$ is any function $F : [a,b] \to \mathbb{R}$ which is continuous on $[a,b]$, differentiable on (a,b) and s.t.

$$\forall x \in (a,b) : \quad F'(x) = f(x)$$

Lemma If $F, \tilde{F}$ are antiderivatives on $[a,b]$ then

$$\forall x \in [a,b] : \quad \tilde{F}(x) - F(x) = \tilde{F}(a) - F(a)$$

Pf: $\tilde{F}(x) - F(x) - \left(\tilde{F}(a) - F(a)\right) \underset{\exists z \in (a,x)}{\overset{\text{MVT}}{=}} \tilde{F}'(z) - F'(z) = f(z) - f(z) = 0$

$\boxtimes$

Def Newton's integral of f on $[a,b]$ is the difference $F(b)-F(a)$ for any antiderivative F of f on $[a,b]$

$$N\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b)-F(a)$$

(whenever F exists)

Then

$$\int_a^b F'(x) = F(b)-F(a) \qquad (\text{FTC I})$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}\big(F(x)-F(a)\big) = F'(x) = f(x)$$

$$(\text{FTC II})$$

Riemann's theory

Q: What's the regularity of $x \mapsto \int_a^x f(t)\,dt$

Lemma Let f be RI on $[a,b]$. Then $F(x) := \int_a^x f(t)\,dt$ is Lipschitz cont.

Pf Let $a \le x < y \le b$. Then

$$F(y)-F(x) = \int_x^y f(t)\,dt$$

So

$$|F(y)-F(x)| \le \left|\int_x^y f(t)\,dt\right| \le \Big(\sup_{t\in[a,b]} |f(t)|\Big)|y-x| \qquad \boxtimes$$

Ex $|x| = \int_0^x [1_{(\ldots}$ …

Lemma Let f be RI …

Then $\forall x \in (a,b)$:

f cont. at $x \Rightarrow F$ …

Pf $(y>x)$ $F(y)-F(x)-f(x)$ …

So $\left|\frac{F(y)-F(x)}{y-x} - f(x\ldots\right.$ …

f RC at $x \Rightarrow \sup_{t\in[\ldots}$ …

Same for left limits.

f on $[a,b]$

$F(a)$ for

f on $[a,b]$

$= F(b) - F(a)$

$F(a)$ (FTC I)

$(a) - F(a)) = F'(x) = f(x)$

(FTC II)

Riemann's theory

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So

$$|F(y)-F(x)| \le \left|\int_x^y f(t)\,dt\right| \le \left(\sup_{t\in[a,b]} |f(t)|\right)|y-x|$$

$\boxtimes$

Ex $|x| = \int_0^x [1_{(0,\infty)}(t) - 1_{(-\infty,0)}(t)]\,dt$

Let f be RI on $[a,b]$, $F(x) := \int_a^x f(t)\,dt$.

Lemma Then $\forall x \in (a,b)$:
f cont. at $x \Rightarrow F'(x)$ exists $\wedge$ $F'(x) = f(x)$

Pf $(y>x)$

$$F(y)-F(x)-f(x)(y-x) = \int_x^y (f(t)-f(x))\,dt$$

So

$$\left|\frac{F(y)-F(x)}{y-x} - f(x)\right| \le \frac{1}{y-x}\left(\sup_{t\in[x,y]} |f(t)-f(x)|\right)(y-x)$$

f RC at $x \Rightarrow \sup_{t\in[x,y]} |f(t)-f(x)| \xrightarrow[y\to x^+]{} 0$

Same for left limits.

$\boxtimes$

Thm (FTC I) Let $f:[a,b]\to\mathbb{R}$ be continuous.
Then $x \longmapsto \int_a^x f(t)\,dt$ is differentiable on (a,b)
and $\forall x\in(a,b)$: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

Remarks:
- $\frac{d}{dx}\int_a^x f(t)\,dt \Big|_{x=a^+} = f(a)$

  and similarly for left derivative at b.
- generalizes to $x \longmapsto \int_x^b f(t)\,dt$

  So $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$

Note Continuity of integrand NOT necessary for differentiability of integral:

Ex $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$

$$f(x) := \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{n+1} & x = q_n \quad (n \in \mathbb{N}) \end{cases}$$

Then $\forall\, a < x$: $\int_a^x f(t)\,dt = 0$.

Ex $$f(x) = \frac{d}{dx} x^2 \sin(1/x) = \begin{cases} -\cos(1/x) + 2x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$\int_0^x f(t)\,dt = x^2 \sin(1/x)$$

Thm (FTC II) Let $F : [a,b] \to \mathbb{R}$ be cont on $[a,b]$, diff on (a,b). Then

$$F' \text{ RI on } [a,b] \Rightarrow \int_a^b F'(x)\,dx = F(b) - F(a)$$

Pf: Given $\varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall \Pi = (\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$ marked partition of $[a,b]$:

$$\|\Pi\| < \delta \Rightarrow \left| R(F', \Pi) - \int_a^b F'(x)\,dx \right| < \varepsilon$$

Take $n := \left\lceil \frac{b-a}{\delta} \right\rceil$, set $t_i := a + \frac{b-a}{n} i$ $(i = 0 \dots n)$

Use **MVT**: $\forall i = 1 \dots n$ $\exists t_i^* \in (t_{i-1}, t_i)$:

$$F(t_i) - F(t_{i-1}) = F'(t_i^*)(t_i - t_{i-1})$$

Then

$$F(b) - F(a) = \sum_{i=1}^n (F(t_i) - F(t_{i-1}))$$
$$= \sum_{i=1}^n F'(t_i^*)(t_i - t_{i-1})$$
$$= R(F', \Pi)$$

So $\left| F(b) - F(a) - \int_a^b F'(x)\,dx \right| < \varepsilon$

and so $F(b) - F(a) = \int_a^b F'(x)\,dx$. $\boxtimes$

Consequences of F

Thm (Integration by

Let $f, g : [a,b] \to \mathbb{R}$

on (a,b) with f', g'

Then $\int_a^b f'(x) g(x)\,dx =$

Pf: $(f \cdot g)'(x) = f'(x)$

is RI by assumptio

So plug for F

II) Let $F:[a,b]\to\mathbb{R}$ be ...b], diff on (a,b). Then

... on $[a,b] \Rightarrow \int_a^b F'(x)\,dx = F(b)-F(a)$

$\varepsilon>0$ $\exists\delta>0$ s.t.

$\Pi = (\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$ marked partition of $[a,b]$:

$\|\Pi\|<\delta \Rightarrow \left|R(F',\Pi) - \int_a^b F'(x)dx\right| < \varepsilon$

$n := \left\lceil \frac{b-a}{\delta} \right\rceil$, set $t_i := a + \frac{b-a}{n} i$ $(i=0\ldots n)$

MVT: $\forall i=1\ldots n$ $\exists t_i^* \in (t_{i-1}, t_i)$:

$F(t_i)-F(t_{i-1}) = F'(t_i^*)(t_i - t_{i-1})$

Then

$$F(b)-F(a) = \sum_{i=1}^n (F(t_i)-F(t_{i-1}))$$
$$= \sum_{i=1}^n F'(t_i^*)(t_i - t_{i-1})$$
$$= R(F', \Pi)$$

So $$\left| F(b)-F(a) - \int_a^b F'(x)dx \right| < \varepsilon$$

and so $F(b)-F(a) = \int_a^b F'(x)dx$. $\boxtimes$

Consequences of FTC's

Thm (Integration by parts)

Let $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b) with f',g' RI on $[a,b]$. (Choose $f'(a), g'(a), f'(b), g'(b)$ arbitrarily)

Then $$\int_a^b f'(x)g(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\,dx$$

Pf: $(f\cdot g)'(x) = f'(x)g(x) + g'(x)f(x)$

is RI by assumptions.

So plug for $F(x) = (f\cdot g)(x)$ into FTC II. $\boxtimes$

<u>Thm</u> (Sub rule) Let $f : [c,d] \to \mathbb{R}$, $\varphi : [a,b] \to (c,d)$.

be s.t.

(1) φ cont on $[a,b]$, diff on (a,b)

(2) f cont. on $[c,d]$

(3) $(f \circ \varphi) \cdot \varphi'$ RI on $[a,b]$.

Then $$\int_{\varphi(a)}^{\varphi(b)} f(t)\,dt = \int_a^b (f \circ \varphi)(x)\,\varphi'(x)\,dx$$

Pf $F(x) := \int_c^x f(t)\,dt$, then $F'(t) = f(t)$ $t \in (c,d)$

$(F \circ \varphi)'(x) = F'(\varphi(x))\,\varphi'(x) = f(\varphi(x))\,\varphi'(x)$

So $\int_a^b f \circ \varphi(x)\,\varphi'(x)\,dx = F \circ \varphi(x)\Big|_a^b = \int_{\varphi(a)}^{\varphi(b)} f(t)\,dt.$ $\boxtimes$

Thm (Taylor's Thm with remainder)

Let $f:(a,b)\to\mathbb{R}$ be $(n+1)$-times diff with $f^{(n+1)}$ RI on any closed subinterval on (a,b). Then

$$\forall\, x, x_0\in(a,b):\quad f(x) = \overbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}^{(\text{Taylor polynomial})\ P_n(x)} + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(z)\,(x-z)^n\,dz$$

Pf by induction:

n=0: by FTC II:

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(z)\,dz$$

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n-general: Assume holds for n. Let f be s.t. $f^{(n+2)}$ is RI on subintervals of (a,b).

Then

$$\frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(z)(x-z)^n\,dz \overset{IBP}{=} \frac{-1}{(n+1)!} f^{(n+1)}(z)(x-z)^{n+1}\Big|_{x_0}^{x} + \frac{1}{(n+1)!}\int_{x_0}^{x} f^{(n+2)}(z)(x-z)^{n+1}\,dz$$

$$\frac{1}{(n+1)!} f^{(n+1)}(z)(x-x_0)^{n+1} = P_{n+1}(x) - P_n(x)$$

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