Riemann/Darboux integral, etc

Last time: f RI on $[a,b] \iff f$ DI on $[a,b]$

$$\iff \boxed{\forall \varepsilon > 0\ \exists\ \Pi = \{t_i\}_{i=0}^{n} \text{ partition of } [a,b]: \quad (*) \\ \sum_{i=1}^{n} \operatorname{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) < \varepsilon}$$

Remains to show: integrals are the same

Let $\{\Pi_n\}_{n\in\mathbb{N}}$ be marked partitions s.t. $U(f,\Pi_n) - L(f,\Pi_n) \to 0$

May assume $\|\Pi_n\| \to 0$. Choosing subsequence, we may assume

$\lim\limits_{n\to\infty} U(f,\Pi_n) = \lim\limits_{n\to\infty} L(f,\Pi_n)$ exist. (and equals Darboux integrals).

But $L(f,\Pi_n) \le R(f,\Pi_n) \le U(f,\Pi_n)$ means $R(f,\Pi_n) \to$ same limit.

$\boxtimes$

Consequences:

- easier proof of additivity (w.r.t. domain of integration)
- extension of DI/RI to other functions

Lemma Let $f, g : [a,b] \to \mathbb{R}$ be bdd.

Then

(1) f RI on $[a,b]$ $\Rightarrow$ $|f|$ RI on $[a,b]$ $\wedge$ $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$

(2) f, g RI on $[a,b]$ $\Rightarrow$ $f \cdot g$ RI on $[a,b]$.

Pf of (1): $\big|\,|f(y)| - |f(x)|\,\big| \leq |f(x) - f(y)|$

$\Rightarrow \operatorname{osc}(|f|, A) \leq \operatorname{osc}(f, A)$

So $(*)$ for f $\Rightarrow$ $(*)$ for $|f|$.

Pf of (2):

$|f \cdot g(y) - f \cdot g(x)|$

$\leq |f(y)|\,|g(y) - g(x)| + |g(x)|\,|f(y) - f(x)|$

$\Rightarrow \operatorname{osc}(f \cdot g, A)$

$\leq \left(\sup_{x \in A} |g(x)|\right) \operatorname{osc}(f, A)$

$+ \left(\sup_{x \in A} |f(x)|\right) \operatorname{osc}(g, A)$ $\boxtimes$

So $(*)$ for f and g $\Rightarrow$ $(*)$ for $f \cdot g$.

Lemm

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Some

Lemma

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(w.r.t. domain of integration)
other functions

be bdd.

on $[a,b]$

$\ldots dx \Big| \le \int_a^b |f(x)|\,dx$

RI on $[a,b]$.

<u>Pf of (1)</u>: $\big||f(y)| - |f(x)|\big| \le |f(x) - f(y)|$

$\Rightarrow \operatorname{osc}(|f|, A) \le \operatorname{osc}(f, A)$

So $(*)$ for $f \Rightarrow (*)$ for $|f|$.

<u>Pf of (2)</u>:

$$|f\cdot g(y) - f\cdot g(x)|$$
$$\le |f(y)|\,|g(y) - g(x)| + |g(x)|\,|f(y) - f(x)|$$

$$\Rightarrow \operatorname{osc}(f\cdot g, A) \le \Big(\sup_{x\in A}|g(x)|\Big)\operatorname{osc}(f,A) + \Big(\sup_{x\in A}|f(x)|\Big)\operatorname{osc}(g,A)$$

So $(*)$ for f and g $\Rightarrow (*)$ for $f\cdot g$. $\boxtimes$

<u>Lemma</u> Let f be <u>RI</u> on $[a,b]$ and $F:\mathbb{R}\to\mathbb{R}$ be s.t. $\overline{\operatorname{Ran}(f)} \subseteq \operatorname{Dom}(F)$ and F cont. on $\overline{\operatorname{Ran}(f)}$. Then $F\circ f$ is RI on $[a,b]$.

Pf HW

Some things are harder for DI!

<u>Lemma</u> <u>Let</u> $f,g : [a,b] \to \underline{\mathbb{R}}$ be bdd. <u>Then</u>

$$\overline{\int_a^b} (f+g)(x)\,dx \le \overline{\int_a^b} f(x)\,dx + \overline{\int_a^b} g(x)\,dx$$

$$\underline{\int_a^b} (f+g)(x)\,dx \ge \underline{\int_a^b} f(x)\,dx + \underline{\int_a^b} g(x)\,dx$$

Pf HW

Sufficient conditions for RI

Lemma Let $f : [a,b] \to \mathbb{R}$. Then

$$\boxed{f \text{ continuous on } [a,b] \Rightarrow f \text{ RI on } [a,b]}$$

Pf $[a,b]$ compact so Bolzano-Weierstass tells us
f continuous $\Rightarrow$ f bdd $\wedge$ f unif. continuous.
So given $\varepsilon > 0$ $\exists \delta > 0$ $\forall s,t \in [a,b]$:

$$0 < t-s < \delta \Rightarrow |f(t) - f(s)| < \frac{\varepsilon}{2(b-a)}$$

So $\forall s,t \in [a,b]$: — " — $\Rightarrow \operatorname{osc}(f, [s,t]) \leq \frac{\varepsilon}{2(b-a)} < \frac{\varepsilon}{b-a}$

So take Π with $\|\Pi\| < \delta$. Then $(*)$ is true!. $\boxtimes$

Lemma Suppose $f:[a,b]\to\mathbb{R}$ is bounded and has finite number of discontinuity points. Then f is RI on $[a,b]$.

Pf Let $x_1,\dots,x_n$ be discontinuities of f. Define, for given $\varepsilon>0$,

$$0<\delta' < \frac{1}{4n}\,\frac{\varepsilon}{1+\sup\limits_{x\in[a,b]}|f(x)|} \;\wedge\; \delta' < \frac{1}{4}\min_{i\neq j}|x_i-x_j|$$

Next set $A := [a,b]\setminus \bigcup_{i=1}^{n}\left(x_i-\frac{\delta'}{2},\, x_i+\frac{\delta'}{2}\right)$

The A is compact so f is unif. cont. on A, so $\exists\,\delta''$ s.t.

$$\forall s,t\in A;\; 0<t-s<\delta'' \wedge [s,t]\subseteq A \Rightarrow \operatorname{osc}(f,[s,t]) < \frac{\varepsilon}{b-a}.$$

Now chose partition Π of $[a,b]$ of $\|\Pi\| < \min\{\delta',\delta''\}$ s.t. the points $[x_i-\delta'/2,\, x_i+\delta'/2]$ belong to Π. Then ...

writing $J = \{ j : [t_{j-1}, t_j] = [x_i - \frac{\delta'}{2}, x_i + \frac{\delta'}{2}]$ for some $i \}$

we have

$$(*) = \sum_{i \in J} \operatorname{osc}(f, [t_i, t_{i-1}])(t_i - t_{i-1})$$

$$+ \sum_{i \notin J} \text{———} \;\; " \;\; \text{———}$$

$$\leq 2(\sup|f|)\,\delta' n$$

$$+ \sum_{i=1}^{n} \frac{\varepsilon}{b-a}(t_i - t_{i-1}) < \varepsilon + \varepsilon = 2\varepsilon. \quad \boxtimes$$

Q: How bad can f be to be RI?

Lemma Let $f(x) = \mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$

Then $\forall a < b$: f NOT RI on $[a,b]$

Pf: $\forall s < t$: $\operatorname{osc}(f, [s,t]) = 1$ $\boxtimes$

Lemma Let $f : [a,b] \to$ be bdd. Then any of

(1) $\forall x \in (a,b)$: $\lim_{z \to x} f(z)$ exists

(2) f has no discontinuities of 2^{nd} kind

(3) $\{x \in (a,b) : f \text{ NOT cont at } x\}$ is finite or countable

implies f is RI on $[a,b]$.

Not end

Lemma L

Then $\mathbb{1}_C$

yet $\mathbb{1}$

Pf HW.

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$= [x_i - \frac{\delta'}{2}, x_i + \frac{\delta'}{2}]$ for some i }

$)(t_i - t_{i-1})$

$t_i - t_{i-1}) < \varepsilon + \varepsilon = 2\varepsilon.$ $\boxtimes$

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Not end of the story:

Lemma Let C = Cantors ternary set.

Then $\mathbb{1}_C$ has discontinuity at all points of C

yet $\mathbb{1}_C$ is RI $\wedge$ $\int_0^1 \mathbb{1}_C(x)\,dx = 0$

Pf HW.

Real characterization needs concept of measure but can be stated and proved without it...

Def $A \subseteq \mathbb{R}$ is said to be of <u>zero length</u> if $\forall \varepsilon > 0$ $\exists \{(a_i, b_i)\}_{i \in \mathbb{N}}$ intervals s.t. $A \subseteq \bigcup_{i \in \mathbb{N}} (a_i, b_i) \wedge \sum_{i \in \mathbb{N}} (b_i - a_i) < \varepsilon$

Thm (Lebesgue/Vitali) Let $f : [a,b] \to \mathbb{R}$ bdd.

Then f RI on $[a,b]$

$\iff \{x \in [a,b] : f \text{ NOT cont at } x\}$ is zero length.