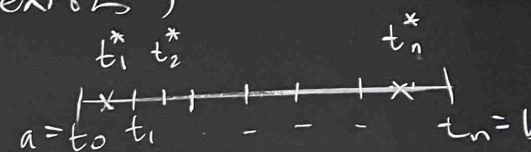
Darboux integral

Assume $f : [a,b] \to \mathbb{R}$ is bounded

Riemann sum

recall $R(f, \Pi) := \sum_{i=1}^{n} f(t_i^*)(t_i - t_{i-1})$

$\int_a^b f(x)\,dx := \lim_{\|\Pi\| \to 0} R(f, \Pi)$ (if exists)



Def Upper Darboux sum is defined as

$$U(f, \Pi) := \sum_{i=1}^{n} \left(\sup_{x \in [t_{i-1}, t_i]} f(x) \right) (t_i - t_{i-1})$$

Lower Darboux sum

$$L(f, \Pi) := \sum_{i=1}^{n} \left(\inf_{x \in [t_{i-1}, t_i]} f(x) \right) (t_i - t_{i-1})$$

Lemma $\forall\, \Pi =$ marked partition of $[a,b]$:

$$L(f,\Pi) \le R(f,\Pi) \le U(f,\Pi)$$

Moreover, $\forall \varepsilon > 0 \ \exists\, \Pi, \Pi' =$ marked partitions s.t.

$$U(f,\Pi) \le R(f,\Pi) + \varepsilon$$

$$L(f,\Pi') \ge R(f,\Pi') - \varepsilon$$

Pf: def of sup/inf.

Lemma Writing $\Pi' \subseteq \Pi$ for the fact that Π is a refinement of Π':

$$\Pi' \subseteq \Pi \Rightarrow L(f,\Pi') \le L(f,\Pi)$$

$$\wedge\ U(f,\Pi) \le U(f,\Pi')$$

In particular,

$$\forall \Pi, \Pi' : \ L(f,\Pi') \le U(f,\Pi)$$

Pf: Assume Π is Π' with point u added to interval $[t_{i-1}, t_i]$

$U(f,\Pi') - U(f,$

$= \left(\sup_{x\in[t_{i-1},t_i]} f(x)\right)($

$- \left(\sup_{x\in[t_{i-1},}\right.$

$- \left(\sup_{x\in[u,t}\right.$

$\ge \left(\sup_{x\in[t_{i-1},t_i]} f(x)\right)[$

Let Π, Π' be partitions

Then $L(f,\Pi') \le L(f,\Pi$

$\le U(f,$

marked Π = partition of $[a,b]$:

$L(f,\Pi) \le R(f,\Pi) \le U(f,\Pi)$

$\forall \varepsilon > 0\ \exists\ \Pi, \Pi'$ = marked partitions

$U(f,\Pi) \le R(f,\Pi) + \varepsilon$

$L(f,\Pi') \ge R(f,\Pi') - \varepsilon$

... of sup/inf.

Lemma: Writing $\Pi' \subseteq \Pi$ for the fact that Π is a refinement of Π':

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In particular,

$$\forall \Pi, \Pi': \ L(f,\Pi') \le U(f,\Pi)$$

Pf: Assume Π is Π' with point u added to interval $[t_{i-1}, t_i]$

$[t_{i-1} \quad u \quad]t_i$

$$U(f,\Pi') - U(f,\Pi)$$

$$= \Big(\sup_{x\in[t_{i-1},t_i]} f(x)\Big)(t_i - t_{i-1}) - \Big(\sup_{x\in[t_{i-1},u]} f(x)\Big)(u - t_{i-1}) - \Big(\sup_{x\in[u,t_i]} f(x)\Big)(t_i - u)$$

$$\ge \Big(\sup_{x\in[t_{i-1},t_i]} f(x)\Big)\Big[\underbrace{(t_i - t_{i-1}) - (u - t_{i-1}) - (t_i - u)}_{=0}\Big] = 0$$

Let Π, Π' be partitions, $\Pi \cup \Pi'$ = common refinement.

Then $L(f,\Pi') \le L(f,\Pi\cup\Pi') \le U(f,\Pi\cup\Pi') \le U(f,\Pi)$ ☒

Def Let $f: [a,b] \to \mathbb{R}$ be bdd. Then

$$\overline{\int_a^b} f(x)\,dx := \inf\left\{ U(f,\Pi) : \Pi \text{ partition of } [a,b] \right\}$$

is the **upper Darboux integral**, and

$$\underline{\int_a^b} f(x)\,dx := \sup\left\{ L(f,\Pi) : \Pi \text{ partition of } [a,b] \right\}$$

is the **lower Darboux integral**.

Corollary $\forall f : [a,b] \to \mathbb{R}$ bounded:

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$$

Def $f : [a,b] \to \mathbb{R}$ bounded is <u>Darboux integrable</u> (DI) on $[a,b]$ if

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

We call common value the <u>Darboux integral</u>

GOAL: Show equivalence of RI & DI

Lemma (Characterization of DI)

For $f : [a,b] \to \mathbb{R}$ bounded:

f DI on $[a,b]$

$\iff \forall \varepsilon > 0 \; \exists \Pi =$ partition of $[a,b]$:

$$U(f,\Pi) - L(f,\Pi) < \varepsilon$$

Pf $\boxed{\Leftarrow}$ $\forall \Pi =$ partitions:

$$0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \le U(f,\Pi) - L(f,\Pi)$$

So $U(f,\Pi) - L(f,\Pi) < \varepsilon \Rightarrow$ integrals differ by $< \varepsilon$.

$\boxed{\Rightarrow}$ Def of inf/sup:

$\exists \Pi'$: $\overline{\int_a^b} f(x)\,dx \ge U(f,\Pi') - \varepsilon$

$\exists \Pi''$: $\underline{\int_a^b} f(x)\,dx \le L(f,\Pi'') + \varepsilon$

Set $\Pi = \Pi' \cup \Pi''$ (common refinement)

Then:

$U(f,\Pi) -$

$\le U($

$\le \int_a$

if f is DI

Corollary: $\forall f$:

$\overline{\int_a^b} f(x)\,dx -$

$= \inf$

...ence of RI & DI

...tion of DI)

...ounded:

Π = partition of $[a,b]$:

$(f,\Pi) - L(f,\Pi) < \varepsilon$

Pf $\boxed{\Leftarrow}$ $\forall \Pi$ = partitions:

$0 \le \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)\,dx \le U(f,\Pi) - L(f,\Pi)$

So $U(f,\Pi) - L(f,\Pi) < \varepsilon \Rightarrow$ integrals differ by $< \varepsilon$.

$\boxed{\Rightarrow}$ Def of inf/sup:

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Set $\Pi = \Pi' \cup \Pi''$ (common refinement)

Then:

$U(f,\Pi) - L(f,\Pi)$

$\le U(f,\Pi') - L(f,\Pi'')$

$\le \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x) + 2\varepsilon$

if f is DI, then RHS $= 2\varepsilon$. $\boxtimes$

Corollary: $\forall f : [a,b] \to \mathbb{R}$ bounded

$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)\,dx$

$= \inf\{ U(f,\Pi) - L(f,\Pi) ;\ \Pi = \text{partition} \}$

Recall: $\operatorname{osc}(f, A) := \sup\{ |f(y) - f(x)| : x, y \in A \}$

Note $\operatorname{osc}(f, A) = \sup_{x \in A} f(x) - \inf_{x \in A} f(x)$

So $U(f, \Pi) - L(f, \Pi)$

$$= \sum_{i=1}^{n} \operatorname{osc}(f, [t_{i-1}, t_i])\,(t_i - t_{i-1})$$

Thm (Equivalence of DI & RI) Let $f : [a,b] \to \mathbb{R}$ be bounded.

Then

$$f \text{ RI on } [a,b] \iff \forall \varepsilon > 0 \ \exists \Pi = \{t_i\}_{i=0}^{n} \text{ partition of } [a,b]: \sum_{i=1}^{n} \operatorname{osc}(f, [t_{i-1}, t_i])(t_i - t_{i-1}) < \varepsilon$$

In particular, f RI $\iff$ f DI

and if both TRUE, then integrals are equal.

Pf $(RI \Rightarrow DI)$

If f RI $\Rightarrow \forall \varepsilon > 0\ \exists \delta > 0\ \forall \Pi, \Pi'$:

$$\|\Pi\|, \|\Pi'\| < \delta \Rightarrow |R(f,\Pi) - R(f,\Pi')| < \varepsilon.$$

Let $n := \left\lceil \frac{b-a}{\delta} \right\rceil$, set $t_i = a + \frac{b-a}{n} i$, $i = 0, \dots, n$.

For each $i = 1, \dots, n$, let $t_i^*, \tilde{t}_i^*$ be s.t.

$$f(t_i^*) \geq \sup_{x \in [t_{i-1}, t_i]} f(x) - \frac{\varepsilon}{b-a}$$

$$f(\tilde{t}_i^*) \leq \inf_{x \in [t_{i-1}, t_i]} f(x) + \frac{\varepsilon}{b-a}$$

Let $\left.\begin{aligned} \Pi &:= \left(\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n \right) \\ \Pi' &:= \left(\text{---}\,\text{''}\,\text{---}, \{\tilde{t}_i^*\}_{i=1}^n \right) \end{aligned}\right\} \|\Pi\|, \|\Pi'\| < \delta$

Then $R(f,\Pi) \geq U(f,\Pi) - \varepsilon$

$R(f,\Pi') \leq L(f,\Pi) + \varepsilon$

So

$$\begin{aligned} U(f,\Pi) - L(f,\Pi) &\leq R(f,\Pi) - R(f,\Pi') + 2\varepsilon \\ &\underset{\|\Pi\|, \|\Pi'\| < \delta}{\leq} \varepsilon + 2\varepsilon = 3\varepsilon \quad \boxtimes \end{aligned}$$

Pf $(DI \Rightarrow RI)$

f DI $\Rightarrow \exists \Pi_0 : U(f,\Pi_0) - L(f,\Pi_0) < \varepsilon$

Let $n = \#$ of partition points of Π_0

Pick $\delta > 0$ s.t.

Let $\Pi, \Pi' =$ ma

Let $\Pi_0 \cup \Pi =$

Claim $|R(f, \Pi_0 \cup \Pi$

Pf Add point u

$|R(f, \Pi \cup \{u$

$\leq 2 \sup_{x \in}$

$-R(f,\Pi')| < \varepsilon$

$\frac{-a}{n} i,\ i=0,\dots,n$

$\|\Pi\|, \|\Pi'\| < \delta$

Then $R(f,\Pi) \geq U(f,\Pi) - \varepsilon$

$R(f,\Pi') \leq L(f,\Pi') + \varepsilon$

So
$$U(f,\Pi) - L(f,\Pi) \leq R(f,\Pi) - R(f,\Pi') + 2\varepsilon \underset{\|\Pi\|,\|\Pi'\|<\delta}{\leq} \varepsilon + 2\varepsilon = 3\varepsilon \qquad \boxtimes$$

Pf ($DI \Rightarrow RI$)

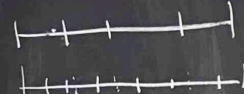
$f\ DI \Rightarrow \exists \Pi_0 :\ U(f,\Pi_0) - L(f,\Pi_0) < \varepsilon$

Let n = # of partition points of Π_0

Pick $\delta > 0$ s.t. $2n\delta\left(\sup_{x\in[a,b]} |f(x)|\right) < \varepsilon$.

Let Π, Π' = marked partitions with $\|\Pi\|, \|\Pi'\| < \delta$.

Let $\Pi_0 \cup \Pi$ = common refinement of Π, Π_0 keep marked pts of Π when possible, o/w choose left endpoint



Claim $\left|R(f,\Pi_0\cup\Pi) - R(f,\Pi)\right| \leq 2n\left(\sup_{x\in[a,b]}|f(x)|\right)\|\Pi\|$ $< \varepsilon$

Pf Add point u to Π, falling into $[t_{i-1}, t_i]$.

$$\left|R(f,\Pi\cup\{u\}) - R(f,\Pi)\right| \leq 2 \sup_{x\in[t_i,t_{i-1}]} |f(x)| (t_i - t_{i-1})$$

Then

$$\left| R(f,\Pi) - R(f,\Pi') \right|$$

$$\leq 2\varepsilon + \left| \underbrace{R(f,\Pi\cup\Pi_0) - R(f,\Pi'\cup\Pi_0)}_{\text{assume } \geq 0} \right|$$

$$\leq 2\varepsilon + \overline{U}(f,\Pi\cup\Pi_0) - L(f,\Pi'\cup\Pi_0)$$

$$\leq 2\varepsilon + \overline{U}(f,\Pi_0) - L(f,\Pi_0)$$

$$\leq 3\epsilon .$$

So DI $\Rightarrow$ RI. $\boxtimes$