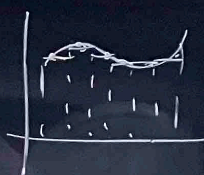
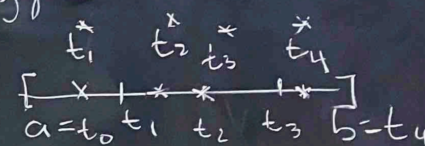
The Riemann integral



Motivation: "area bounded by curve"

Solutions: 500 BC exhaustion by polygons

Real def. B. Riemann 1854



Def: Let $a<b$. We say:

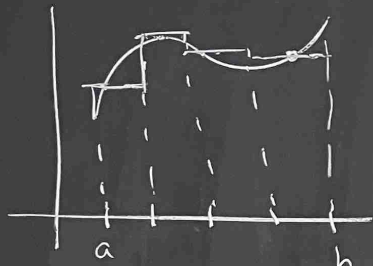
- A marked partition of $[a,b]$ Π is pair $\left(\{t_i\}_{i=0}^{n}, \{t_i^*\}_{i=1}^{n}\right)$ s.t.

  $a=t_0<t_1<\ldots<t_n=b \wedge \forall i=1\ldots n: t_i^* \in [t_{i-1}, t_i]$

- The mesh of Π is $\|\Pi\| := \max\limits_{i=1\ldots n} |t_i - t_{i-1}|$

- Given $f : [a,b] \to \mathbb{R}$, its Riemann sum associated with $\Pi = (\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$ is

$$R(f, \Pi) := \sum_{i=1}^{n} f(t_i^*)(t_i - t_{i-1})$$



interpretation as "signed" area.

"RI on $[a,b]$"

Def $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ if $\exists L \in \mathbb{R}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall \Pi$ = marked partition of $[a,b]$:

$$\|\Pi\| < \delta \Rightarrow |R(f, \Pi) - L| < \varepsilon$$

Notation: $\lim\limits_{\|\Pi\| \to 0} R(f, \Pi) = L$

Lemma If L in the def of RI exists, then it is unique.

Pf Let def hold with L and L'. Given $\varepsilon > 0$, let δ and δ' be as in the def, respectively. Then $\forall \Pi$: $\|\Pi\| < \min\{\delta, \delta'\}$:

$$|L - L'| \le |R(f, \Pi) - L| + |R(f, \Pi) - L'|$$
$$\le \varepsilon + \varepsilon = 2\varepsilon$$

So $L' = L$. $\boxtimes$

Def If f: [...]
we set $\int_a^b f$ [...]
and call this [...]

Lemma (Cauchy [...])
Let $f : [a,b] \to \mathbb{R}$ [...]
f RI on $[a,b]$ [...]

Pf HW

..., its Riemann sum

$(\{t_i\}_{i=0}^n, \{t_i^*\}_{i=1}^n)$

$\sum_{i=1}^n f(t_i^*)(t_i - t_{i-1})$

...retation as "signed" area.

"RI on $[a,b]$"

...iemann integrable on $[a,b]$

...Π = marked partition of $[a,b]$:

$\Rightarrow |R(f,\Pi) - L| < \varepsilon$

Notation: $\lim_{\|\Pi\| \to 0} R(f,\Pi) = L$

Lemma If L in the def of RI exists, then it is unique.

Pf Let def hold with L and L'. Given $\varepsilon > 0$, let δ and δ' be as in the def, respectively. Then $\forall \Pi : \|\Pi\| < \min\{\delta, \delta'\}$:

$$|L - L'| \le |R(f,\Pi) - L| + |R(f,\Pi) - L'|$$
$$\le \varepsilon + \varepsilon = 2\varepsilon$$

So $L' = L$. $\boxed{\times}$

Def If $f : [a,b] \to \mathbb{R}$ is RI on $[a,b]$ we set

$$\int_a^b f(x)\,dx = \lim_{\|\Pi\| \to 0} R(f,\Pi)$$

and call this the Riemann integral of f from a to b.

Lemma (Cauchy criterion for RI)
Let $f : [a,b] \to \mathbb{R}$. Then

$$f \text{ RI on } [a,b] \iff \forall \varepsilon > 0\ \exists \delta > 0\ \forall \Pi, \Pi' :$$
$$\|\Pi\|, \|\Pi'\| < \delta \Rightarrow |R(f,\Pi) - R(f,\Pi')| < \varepsilon$$

Pf HW

Lemma (RI $\Rightarrow$ boundedness) Let $f : [a,b] \to \mathbb{R}$.

Then f RI on $[a,b]$ $\Rightarrow$ $\sup_{x \in [a,b]} |f(x)| < \infty$.

Moreover, f RI on $[a,b]$ implies

$$\left| \int_a^b f(x)\,dx \right| \leq \left(\sup_{x \in [a,b]} |f(x)| \right)(b-a)$$

Pf: Let $\varepsilon := 1$, $\delta > 0$ as in def of RI.

Set $n \in \mathbb{N}$ s.t. $n\delta > b-a$ $\quad \left(n := \left\lceil \frac{b-a}{\delta} \right\rceil\right)$

Let $t_j := a + \frac{b-a}{N} j$, $\quad j = 0, \dots, n$

For $k = 1 \dots, N$ set. $t_j^* = \begin{cases} t_{j-1} & j \neq k \\ t & j = k \end{cases} \quad t \in [t_{k-1}, t_k]$.

Set $\Pi = \left(\{t_i\}_{i=0}^{n}, \{t_i^*\}_{i=1}^{n}\right)$. Then

$$f(t)(t_k - t_{k-1}) = R(f, \Pi) - \sum_{\substack{j=1 \\ j \neq k}}^{n} f(t_j^*)(t_j - t_{j-1})$$

So
$$|f(t)| \underbrace{(t_k - t_{k-1})}_{\frac{b-a}{N}} \leq \left|\int_a^b f(x)\,dx\right| + 1 + \sum_{j=1}^{n} |f(t_{j-1})|(t_j - t_{j-1})$$

Optimizing over t and k we get

$$\sup_{x \in [a,b]} |f(x)| \leq \frac{N}{b-a}\left[\qquad\right] < \infty. \quad \boxtimes$$

Properties of RI

Lemma Let f, g be RI on $[a,b]$. Let $\alpha, \beta \in \mathbb{R}$.
Then $\alpha f + \beta g$ RI on $[a,b]$ and

$$\int_a^b (\alpha f + \beta g)(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

Pf Given $\varepsilon > 0$, find δ' s.t. $\|\Pi\| < \delta' \Rightarrow \left| R(f,\Pi) - \int_a^b f(x)\,dx \right| < \frac{\varepsilon}{1+|\alpha|+|\beta|}$

Similarly, find δ'' s.t. $\|\Pi\| < \delta''$
$\Rightarrow \left| R(g,\Pi) - \int_a^b g(x)\,dx \right| < \frac{\varepsilon}{1+|\alpha|+|\beta|}$

Key point: $R(\alpha f + \beta g, \Pi) = \alpha R(f,\Pi) + \beta R(g,\Pi)$

So $\forall\, \Pi$ s.t. $\|\Pi\| < \min\{\delta', \delta''\}$:

$$\left| R(\alpha f + \beta g, \Pi) - \alpha \int_a^b f(x)\,dx - \beta \int_a^b g(x)\,dx \right|$$

$$= \left| \alpha \left(R(f,\Pi) - \int_a^b f(x)\,dx \right) + \beta \left(R(g,\Pi) - \int_a^b g(x)\,dx \right) \right|$$

$$\leq |\alpha| \left| R(f,\Pi) - \int_a^b f(x)\,dx \right| + |\beta| \left| R(g,\Pi) - \int_a^b g(x)\,dx \right|$$

$$\leq |\alpha| \frac{\varepsilon}{1+|\alpha|+|\beta|} + |\beta| \frac{\varepsilon}{1+|\alpha|+|\beta|} < \varepsilon \quad \boxtimes$$

Lemma
Thm
$(\forall x$

Pf Not

So we lear
$\mathcal{R} := \{ f$
$\varphi : \mathcal{R} \to$
φ is bou
φ is pos

$[a,b]$. Let $\alpha,\beta \in \mathbb{R}$.
$b]$ and
$\ldots dx + \beta \int_a^b g(x)\,dx$

$\delta' \Rightarrow \left| R(f,\Pi) - \int_a^b f(x)\,dx \right| < \frac{\varepsilon}{1+|\alpha|+|\beta|}$

$\left| R(g,\Pi) - \int_a^b g(x)\,dx \right| < \frac{\varepsilon}{1+|\alpha|+|\beta|}$

$) = \alpha R(f,\Pi) + \beta R(g,\Pi)$

So Π s.t. $\|\Pi\| < \min\{\delta',\delta''\}$:

$$\left| R(\alpha f + \beta g, \Pi) - \alpha \int_a^b f(x)\,dx - \beta \int_a^b g(x)\,dx \right|$$

$$= \left| \alpha\left(R(f,\Pi) - \int_a^b f(x)\,dx \right) + \beta\left(R(g,\Pi) - \int_a^b g(x)\,dx \right) \right|$$

$$\leq |\alpha| \left| R(f,\Pi) - \int_a^b f(x)\,dx \right| + |\beta| \left| R(g,\Pi) - \int_a^b g(x)\,dx \right|$$

$$\leq |\alpha| \frac{\varepsilon}{1+|\alpha|+|\beta|} + |\beta| \frac{\varepsilon}{1+|\alpha|+|\beta|} < \varepsilon$$

$\boxtimes$

Lemma (Monotonicity) Let f, g be RI on $[a,b]$.
Then
$$\left(\forall x \in [a,b] : f(x) \leq g(x) \right) \Longrightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

Pf Note $R(f,\Pi) \leq R(g,\Pi)$ & take limit.

So we learned that:

- $\mathcal{R} := \{ f \in \mathbb{R}^{[a,b]} : \text{RI on } [a,b] \}$ is a vector space
- $\varphi : \mathcal{R} \to \mathbb{R}$ defined by $\varphi(f) := \int_a^b f(x)\,dx$ is linear functional
- φ is bounded relative to sup-norm
- φ is positive

Lemma (Additivity) Let $a < c < b$, $f : [a,b] \to \mathbb{R}$.

$$f \text{ RI on } [a,b] \iff f \text{ RI on } [a,c] \;\wedge\; f \text{ RI on } [c,b]$$

and when both TRUE,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Pf of $\Rightarrow$

Assume f RI on $[a,b]$.

So given $\varepsilon>0$ $\exists\delta>0$ s.t. $\forall \Pi$ = part. of $[a,b]$

$\|\Pi\|<\delta \Rightarrow \left| R(f,\Pi) - \int_a^b f(x)\,dx \right| < \varepsilon$.

Assume Π_1 = partition of $[a,c]$; Π_2, Π_2' = part. of $[c,b]$.

Def $\Pi = \Pi_1 \circ \Pi_2$, $\Pi' = \Pi_1 \circ \Pi_2'$ .. Key fact.

$$R(f,\Pi_2) - R(f,\Pi_2') = R(f,\Pi) - R(f,\Pi')$$

So $\|\Pi_1\|, \|\Pi_2\|, \|\Pi_2'\| < \delta \Rightarrow |RHS| \leq 2\varepsilon$ So.

$\forall \Pi_2, \Pi_2'$ = part. of $[c,b]$; $\|\Pi_2\|, \|\Pi_2'\| < \delta \Rightarrow |R(f,\Pi_2) - R(f,\Pi_2')| < \varepsilon$

Cauchy criterion $\Rightarrow$ f is RI on $[b,c]$.

Additivity of integrals follows from add. of Riemann sums.

Pf of $\Leftarrow$:

Assume f RI on $[a,c]$, $[c,b]$.

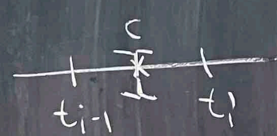
If Π = part. of $[a,b]$ containing c

then $\Pi = \Pi_1 \circ \Pi_2$,

$$R(f,\Pi) = R(f,\Pi_1) + R(f,\Pi_2)$$

Key issue: what if Π doesn't contain c.

Let Π be such partition.



Let Π' be Π with c added and c chosen as marked point.

Then

$$R(f,\Pi') - R(f,\Pi)$$
$$= f(c)(t_i - c) + f(c)(c - t_{i-1}) - f(t_i^*)(t_i - t_{i-1})$$
$$= (f(c) - f(t_i^*))(t_i - t_{i-1})$$

So

$$|R(f,\Pi) - R(f,\Pi')| \le 2\left(\sup_{x\in[a,b]} |f(x)|\right) \|\Pi\|$$

Since $\|\Pi\| \to 0$, we are done. $\boxtimes$