Higher derivatives & multivariate extrema

Lemma Let $f : \mathbb{R}^n \to \mathbb{R}$, $x \in \operatorname{int} \operatorname{Dom}(f)$.
Assume x loc. extremum of f. Then

$$\forall v \in \mathbb{R}^n: \quad \frac{\partial f}{\partial v}(x) \text{ exists} \Rightarrow \frac{\partial f}{\partial v}(x) \geqq 0$$

$$\forall i = 1 \ldots n: \quad \frac{\partial f}{\partial x_i}(x) \text{ exits} \Rightarrow \frac{\partial f}{\partial x_i}(x) = 0$$

In particular, if f diff. at x, then

$$Df(x) = \nabla f(x) = 0$$

Pf x loc. min $\Rightarrow \exists \delta > 0 \ \forall z \in B(x,\delta): f(z) \geq f(x)$

So if $v \in \mathbb{R}^n$, then

$$f(x+hv) \geq f(x)$$

for $h>0$ small ($h\|v\| < \delta$)

So $$\frac{\partial f}{\partial v}(x) = \lim_{h \to 0^+} \frac{f(x+hv) - f(x)}{h} \geq 0.$$

If $\frac{\partial f}{\partial x_i}$ exists, then

$$\frac{\partial f}{\partial x_i}(x) = \underbrace{\frac{\partial f}{\partial e_i}(x)}_{\geq 0} = \underbrace{- \frac{\partial f}{\partial (-e_i)}(x)}_{\leq 0} = 0$$

$\boxtimes$

Ex $f(x,y) = x^2 + y^2$

$$\nabla f(x,y) = (2x, 2y) \overset{(x,y)=0}{=} (0,0)$$

Ex $f(x,y) = \sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial v}(0,0) = \|v\| \geq 0.$$

Further analysis requires higher derivatives

Def We say f is twice differentiable at x if Df is differentiable at x.

We write $D^2 f(x) := D(Df)(x)$

Def The k-th

is defined

Given $i_1, \ldots$

$$\frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}$$

This the k-th

w.r.t. variables

Ex: $f(x,y) =$

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

$\exists \delta > 0 \ \forall z \in B(x,\delta).\ f(z) \geq f(x)$

..., then

$...) \geq f(x)$

$(\ h \ \|h v\| < \delta)$

$...) = \lim_{h \to 0^+} \frac{f(x+hv) - f(x)}{h} \geq 0.$

...atives, then

$\frac{\partial f}{\partial x_i}(x) = \underbrace{\frac{\partial f}{\partial e_i}(x)}_{\geq 0} = - \underbrace{\frac{\partial f}{\partial (-e_i)}(x)}_{\leq 0} = 0$

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Ex $f(x,y) = x^2 + y^2$

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Ex $f(x,y) = \sqrt{x^2+y^2}$

$\frac{\partial f}{\partial v}(0,0) = \|v\| > 0.$

Further analysis requires higher derivatives

Def We say f is twice differentiable at x if Df is differentiable at x.
We write $D^2 f(x) := D(Df)(x)$

Def The k-th-partial derivative of $f: \mathbb{R}^n \to \mathbb{R}$ is defined recursively as:

Given $i_1, \dots, i_k \in \{1, \dots, n\}$,

$$\frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}} := \frac{\partial}{\partial x_{i_1}} \left(\frac{\partial^{k-1} f}{\partial x_{i_2} \dots \partial x_{i_k}} \right)$$

This the k-th partial derivative of f w.r.t. variables $x_{i_1} \dots x_{i_k}$.

Ex: $f(x,y) = x^4 + y^3 x$, $\frac{\partial f}{\partial x} = 4x^3 + y^3$, $\frac{\partial f}{\partial y} = 3y^2 x$

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 12x^2 & 3y^2 \\ 3y^2 & 6yx \end{pmatrix}$$

Thm (Clairaut's thm, proved by KH Schwarz 1873)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $x \in \operatorname{int} \operatorname{Dom}(f)$ be s.t. for some $\delta > 0$:

- f cont. on $B(x, \delta)$
- $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}$ exist and are cont. on $B(x, \delta)$
- $\frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial^2 f}{\partial x_2 \partial x_1}$ exist on $B(x, \delta)$

Then $\frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial^2 f}{\partial x_2 \partial x_2}$ cont at $x \Rightarrow \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(x)$

Pf $h(t,s) = f(x+te_1+se_2) - f(x+te_1) - f(x+se_1) + f(x)$

MVT: $(t,s \geq 0)$: $h(t,0) = 0$

$\exists s' \in (0,s)$: $h(t,s) = \left(\frac{\partial f}{\partial x_2}(x+te_1+s'e_2) - \frac{\partial f}{\partial x_2}(x+s'e_2)\right) s$

$\exists t' \in (0,t)$: $\overset{\text{MVT}}{=} \frac{\partial^2 f}{\partial x_1 \partial x_2}(x+t'e_1+s'e_2)\, st$

By symmetry:

$\exists s'' \in (0,s)\ \exists t'' \in (0,t)$: $h(t,s) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(x+t''e_1+s''e_2)\, st$

As $t \to 0^+, s \to 0^+$, both derivatives converge to their value at x.

The matrix

$$\mathrm{Hess}(f)(x) := \left\{ \frac{\partial^2 f}{\partial x_i \partial x_j} \right\}_{i,j=1}^{n}$$

is called the <u>Hessian of f at x</u>

Lemma Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable on an open neighborhood of x. Then x is local extremum implies

$$\forall v \in \mathbb{R}^n: \; v \cdot \mathrm{Hess}(f)(x)\, v \geq 0,$$

i.e., $\mathrm{Hess}(f)(x)$ is positive semidefinite.

Pf $h(t) = f(x+tv) - f(x)$

$$\overset{\nabla f(x)=0}{=} f(x+tv) - f(x) - t v \cdot \nabla f(x)$$

$$\lim_{t\to 0} \frac{h(t)}{t^2} = \lim_{t\to 0} \frac{v \cdot \nabla f(x+tv) - v \cdot \nabla f(x)}{t}$$

$$= \lim_{t\to 0} \sum_{i=1}^{n} v_i \frac{\frac{\partial f}{\partial x_i}(x+tv) - \frac{\partial f}{\partial x_i}(x)}{t}$$

$$= \sum_{i,j=1}^{n} v_i v_j \lim_{t\to 0} \frac{\partial^2 f}{\partial x_j \partial x_i}(x+tv)$$

$$= v \cdot \mathrm{Hess}(f)(x)\, v \; \geq 0 \quad \boxtimes$$

x loc. min $\Rightarrow$ $h(t) \geq 0$ for t small $\Rightarrow$

Lemma Let $f: \mathbb{R}^n$ … be s.t. $\exists \delta > 0$: f …

and $\nabla f(x) = 0$

and $\forall z \in B(x,\delta)$:

Then x is local mi…

Pf: Use h above: Ta…
(Assuming $t \geq 0$) $\exists$ …

$\}^n$

$x_j \}_{i,j=1}$

f at x

$\mathbb{R}$ is twice continuously

neighborhood of x

n implies

$v \geq 0$.

e semidefinite.

Pf $h(t) = f(x+tv) - f(x)$

$\overset{\nabla f(x)=0}{=} f(x+tv) - f(x) - t\, v\cdot\nabla f(x)$

$$\lim_{t\to 0} \frac{h(t)}{t^2} = \lim_{t\to 0} \frac{v\cdot\nabla f(x+tv) - v\cdot\nabla f(x)}{t}$$

$$= \lim_{t\to 0} \sum_{i=1}^{n} v_i \frac{\frac{\partial f}{\partial x_i}(x+tv) - \frac{\partial f}{\partial x_i}(x)}{t}$$

$$= \sum_{i,j=1}^{n} v_i v_j \lim_{t\to 0} \frac{\partial^2 f}{\partial x_j \partial x_i}(x+tv)$$

$$= v\cdot \mathrm{Hess}(f)(x)\, v. \quad \geq 0$$

x loc. min $\Rightarrow$ $h(t) \geq 0$ for t small $\Rightarrow$ $\boxtimes$

<u>Lemma</u> Let $f : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathrm{Int\,Dom}(f)$ be s.t. $\exists \delta > 0$: f twice cont. diff on $B(x,\delta)$

and $\nabla f(x) = 0$

and $\forall z \in B(x,\delta)$: $\mathrm{Hess}(f)(z)$ is pos. semi-def.

Then x is local minimum of f.

Pf: Use h above: Taylor's theorem:

(Assuming $t \geq 0$) $\exists t' \in (0,t)$: $h(t) = \frac{1}{2} t^2 h''(t')$

$$= \frac{1}{2} t^2\, v\cdot \mathrm{Hess}(x+t'v)\, v$$

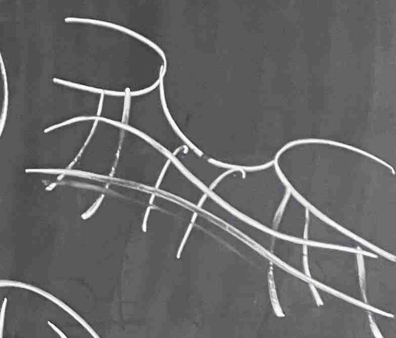
$$\geq 0 \qquad \boxtimes$$

Ex $f(x,y) = x^2 + y^2$

$$\operatorname{Hess}(f)(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Ex $f(x,y) = x^2 + y^2 - 8xy$

$$\operatorname{Hess}(f)(0,0) = \begin{pmatrix} 2 & -8 \\ -8 & 2 \end{pmatrix}$$



Not pos. semidef.

Ex $f(x,y) = -(x^2 + y^2)$