Inverse & implicit function theorems

Last time: Let $f: \mathbb{R}^n \to \mathbb{R}^n$, $\delta_0 > 0$ s.t. $B(x, \delta_0) \subseteq Dom(f)$, and assume f diff. on $B(x, \delta_0)$, Df cont at x, $Df(x)^{-1}$ exists.

Then we showed $\exists \delta > 0$ (s.t. $\|Df(x)^{-1}\| \, \|Df(z) - Df(x)\| \le 1/2 \ \forall z \in B(x,\delta)$)

s.t. 1) f is injective on $B(x,\delta)$ $\wedge$ 2) f is open on $B(x,\delta)$.

Now: This implies that $f^{-1}: f(B(x,\delta)) \to B(x,\delta)$ exists and is continuous on $f(B(x,\delta))$.

Inverse function rule $\Rightarrow$ f^{-1} is differentiable on $f(B(x,\delta))$

$\wedge$ $D(f^{-1})(z) = Df(z)^{-1}$ $\forall z \in f(B(x,\delta))$

Last thing to do: Continuity of $D(f^{-1})$ at $f(x)$:

Recall $\|A^{-1}\|\,\|B\| < 1 \Rightarrow (A+B)^{-1}$ exists

$$\wedge \quad \|(A+B)^{-1} - A^{-1}\| \le \frac{\|A^{-1}\|^2}{\underbrace{1 - \|A^{-1}\|\,\|B\|}_{\ge \frac{1}{2}}}\|B\|$$

Use this for $A := Df(x)$

$B := Df(z) - Df(x)$

to get

$$\|Df(z)^{-1} - Df(x)^{-1}\| \le 2\,\|Df(x)^{-1}\|^2\,\|Df(z) - Df(x)\|$$

Since cont. of Df at x $\Rightarrow$ $\|Df(z) - Df(x)\| \xrightarrow[z \to x]{} 0$

we get $\|D(f^{-1})(y) - D(f^{-1})(f(x))\| \le \|Df(x)^{-1}\|^2\,\|Df(f^{-1}(y)) - Df(x)\|$

RHS tends to zero as $y \to f(x)$ by continuity of f^{-1} which implies $f^{-1}(y) \to f^{-1}(f(x)) = x$ $\boxtimes$

Remarks: Continuity of Df (at x) essential

$n=1$ $\quad f(x) := \begin{cases} x + a x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

then $\quad f'(x) = \begin{cases} 1 - a\cos(1/x) + O(x) \\ 1 \end{cases}$

Not cont once $a \neq 0$.

Not injective near 0 when $|a| > 1$.

- Yet, there is a ver[...] is replaced by Df[...]
- For practical purposes, [...] of Df in open set c[...] This gives continuity [...] (see the text book)
- Key step of proof.
- What drives the proof i[...]

$y = f(x) + Df(x)($[...]

which can be inverted

...g to do: Continuity of $D(f^{-1})$ at $f(x)$:

$\|A^{-1}\|\,\|B\| < 1 \Rightarrow (A+B)^{-1}$ exists

$\wedge\ \|(A+B)^{-1} - A^{-1}\| \leq \dfrac{\|A^{-1}\|^2}{\underbrace{1 - \|A^{-1}\|\,\|B\|}_{\geq \frac{1}{2}}} \|B\|$

... for $A = Df(x)$

$B = Df(z) - Df(x)$

$\|Df(z)^{-1} - Df(x)^{-1}\| \leq 2\,\|Df(x)^{-1}\|^2\,\|Df(z) - Df(x)\|$

...cont. of Df at $x \Rightarrow \|Df(z) - Df(x)\| \xrightarrow[z \to x]{} 0$

$\|D(f^{-1})(y) - D(f^{-1})(f(x))\| \leq \|Df(x)^{-1}\|^2\,\|Df(f^{-1}(y)) - Df(x)\|$

RHS tends to zero as $y \to f(x)$ by continuity of f^{-1} which implies $f^{-1}(y) \to f^{-1}(f(x)) = x$ $\boxtimes$

<u>Remarks:</u> Continuity of $\underset{(\text{at } x)}{Df}$ essential

<u>n=1</u> $f(x) := \begin{cases} x + a x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

then $f'(x) = \begin{cases} 1 - a\cos(1/x) + O(x) \\ 1 \end{cases}$

Not cont once $a \neq 0$.

Not injective near 0 when $|a| > 1$.

- Yet, there is a version where continuity is replaced by Df^{-1} exists on $B(x, \delta_0)$.
- For practical purposes, we usually continuity of Df in open set containing x.

  This gives continuity of Df^{-1} as well (see the text book)

- Key step of proof: f homeomorphic (locally)
- What drives the proof is <u>linear approximation</u>

  $y = f(x) + Df(x)(z - x)$

  which can be inverted: $z = x + Df(x)^{-1}(y - f(x))$

Implicit function theorem

GOAL: Suppose $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ given.

Want to solve for y from eqn's:

$$F_1(x,y)=0 \wedge \dots \wedge F_m(x,y)=0$$

Method of elimination suggest this should work.

Need: $y \longmapsto F(x,y)$ is invertible for given x.

This is what we solved in Inverse function thm.

Thm (Implicit function theorem).

Let $m,n \geq 1$, $F : \mathbb{R}^{m+n} \to \mathbb{R}^m$, $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^m$ s.t. $\underline{\underline{F(x_0,y_0)=0}}$

Assume $\exists\, \delta_0 > 0$ s.t. F is diff. on $B((x_0,y_0),\delta_0)$

with DF continuous at (x_0,y_0) and

$$D_y f(x,y) = \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(x,y) & \cdots & \frac{\partial F_1}{\partial y_m}(x,y) \\ \vdots & \ddots & \\ \frac{\partial F_m}{\partial y_1}(x,y) & \cdots & \frac{\partial F_m}{\partial y_m}(x,y) \end{pmatrix}$$ invertible at (x_0,y_0).

Then $\exists\, \delta > 0$ and function $g : B(x_0,\delta) \to \mathbb{R}^m$ s.t.

$$g(x_0) = y_0 \wedge \forall z \in B(x_0,\delta) : F(x, g(x)) = 0$$

Moreover, g is diff on $B(x,\delta)$ and Dg is cont at x_0.

$\underline{\text{Pf}}$: Def $f : B((x_0, y_0), \delta_0) \to \mathbb{R}^{n+m}$ by

$$f(x,y) = \begin{pmatrix} x \\ F(x,y) \end{pmatrix}$$

$$Df(x,y) = \left(\begin{array}{c|c} I & 0 \\ \hline D_x F_{(x,y)} & D_y F_{(x,y)} \end{array}\right)$$

$\underline{\text{Fact:}}$ DF continuous $\Rightarrow$ Df continuous.

$$(Df)^{-1} = \left(\begin{array}{c|c} I & 0 \\ \hline -(D_y F)^{-1} D_x F & (D_y F)^{-1} \end{array}\right)$$

(at (x_0, y_0))

So f satisfies condition of Inverse function thm!

So f^{-1} exists and is continuous on $f(B((x_0,y_0),\delta'))$ for some $\delta' > 0$

$\underline{\text{Notice}}$ writing $f^{-1}(u,v) = \begin{pmatrix} \tilde{h}(u,v) \\ h(u,v) \end{pmatrix}$ the fact $f \circ f^{-1} = \mathrm{id}$ gives:

$$\begin{pmatrix} u \\ v \end{pmatrix} = f \circ f^{-1}(u,v) = \begin{pmatrix} \hat{h}(u,v) \\ F(u, h(u,v)) \end{pmatrix}$$

So $\hat{h}(u,v) = u \wedge \underline{\underline{F(u, h(u,v)) = v}}$

Note $F(x_0, y_0) = 0$ =

Image open $\Rightarrow \exists \delta > 0$

Now set

$g(x) = h($

and note that

$g(x_0) = y$

and $\forall x \in B(x_0, \delta)$:

Now check diff and co

from same properties o

... by

...ntinuous.

(at (x_0,y_0))

So f satisfies condition of Inverse function thm !.

So f^{-1} exists and is continuous on $f(B((x_0,y_0),\delta'))$ for some $\delta'>0$.

Notice writing $f^{-1}(u,v) = \begin{pmatrix}\tilde{h}(u,v)\\ h(u,v)\end{pmatrix}$ the fact $f \circ f^{-1} = id$ gives:

$$\begin{pmatrix}u\\ v\end{pmatrix} = f\circ f^{-1}(u,v) = \begin{pmatrix}\hat{h}(u,v)\\ F(u,h(u,v))\end{pmatrix}$$

So $\hat{h}(u,v) = u \wedge F(u,h(u,v)) = v$

Note $F(x_0,y_0)=0 \Rightarrow (x_0,0)\in f(B((x_0,y_0)),\delta_0)$

Image open $\Rightarrow \exists \delta>0$: $B(x_0,\delta)\times\{0\} \in f($—''—$)$

Now set

$$g(x) = h(x,0) \qquad x\in B(x_0,\delta)$$

and note that

$$g(x_0) = y_0$$

and $\forall x \in B(x_0,\delta)$: $F(x,g(x)) = 0$

Now check diff. and continuity of Dg at x_0 from same properties of f^{-1}.

$\boxtimes$

$\underline{Ex}$ $\quad xu - yv = 0$

$\quad\quad yu + xv = 1$

$$F(x,y,u,v) = (xu - yv,\ yu + xv - 1)$$

$$D_{(u,v)}F = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

$\det(\quad) = x^2 + y^2 \neq 0$ when $\underline{(x,y) \neq 0}$

Thm applies ... ... $u + x\dfrac{\partial u}{\partial x} - y\dfrac{\partial v}{\partial x} = 0$

$$y\frac{\partial u}{\partial x} + v + x\frac{\partial v}{\partial x} = 0$$