Inverse function theorem

recall $A \in M_{n,n}$, A^{-1} exists, $B \in M_{n,n}$:

$$\|B\|\,\|A^{-1}\| < 1 \Rightarrow (A+B)^{-1} \text{ exists} \wedge \|(A+B)^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|B\|\,\|A^{-1}\|}$$

$$\wedge \; \|(A+B)^{-1} - A^{-1}\| \le \frac{\|A^{-1}\|^2}{1 - \|B\|\,\|A^{-1}\|}\,\|B\|$$

Lemma Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $x \in \operatorname{int} \operatorname{Dom}(f)$ s.t. $f(x) \in \operatorname{int} \operatorname{Ran}(f)$.
Assume f injective, differentiable at x with $Df(x)$ invertible.
and assume f^{-1} continuous at $f(x)$. Then

f^{-1} is differentiable at $f(x)$ $\wedge$ $D(f^{-1})(f(x)) = Df(x)^{-1}$.

Pf Recall $f(z) - f(x) = [Df(x) + u_x(z)](z - x)$

where $\lim_{z \to x} \| u_x(z) \| = 0$

Note: $Df(x)$ invertible $\Rightarrow$

$\exists \delta > 0 \quad \forall x \in B(x, \delta): \quad (Df(x) + u_x(z))^{-1}$ exists

$\wedge \quad \| (Df(x) + u_x(z))^{-1} - Df(x)^{-1} \| < \varepsilon$.

Next: f^{-1} continuous at $f(x)$ means

$\exists \varepsilon > 0: \quad B(f(x), \varepsilon) \subseteq f(B(x, \delta))$

Then for $y = f(z) \in B(f(x), \varepsilon)$

$$y - f(x) = [Df(x) + u_x(z)](f^{-1}(y) - f^{-1}(f(x)))$$

$$f^{-1}(y) - f^{-1}(f(x)) = [Df(x) + u_x(z)]^{-1}(y - f(x))$$

$$= [Df(x)^{-1} + \tilde{u}_{f(x)}(y)](y - f(x))$$

where

$$\tilde{u}_{f(x)}(y) = [Df(x) + u_x(z)]^{-1} - Df(x)^{-1}$$

Now $y \to f(x) \xRightarrow{f^{-1} \text{ cont at } f(x)} z = f^{-1}(y) \to x$

So $\| \tilde{u}_{f(x)}(y) \| \leq \dfrac{\| Df(x)^{-1} \|^2}{1 - \| Df(x)^{-1} \| \| u_x(z) \|} \| u_x(z) \|$

$\Rightarrow \lim_{y \to f(x)} \| \tilde{u}_{f(x)}(y) \| = 0$.

GOAL: f

Thm (In

Let f:

$\exists \delta_0 > 0$

(1) f

(2)

Then we

- f inj
- f^{-1} d
- Df^{-1}

$Df(x) + u_x(z)\big](z-x)$

0

ble $\Rightarrow$

$\delta)$. $(Df(x)+u_x(z))^{-1}$ exists

$)^{-1} - Df(x)^{-1}\| < \varepsilon$.

at $f(x)$ means

$),\varepsilon) \subseteq f(B(x,\varepsilon))$

$B(f(x),\varepsilon)$

$+u_x(z)\big]\big(f^{-1}(y) - f^{-1}(f(x))\big)$

$$f^{-1}(y) - f^{-1}(f(x)) = \big[Df(x) + u_x(z)\big]^{-1}(y - f(x))$$

$$= \big[Df(x)^{-1} + \tilde{u}_{f(x)}(y)\big](y - f(x))$$

where

$$\tilde{u}_{f(x)}(y) = \big[Df(x) + u_x(z)\big]^{-1} - Df(x)^{-1}$$

Now $y \to f(x) \xRightarrow{f^{-1} \text{ cont at } f(x)} z = f^{-1}(y) \to x$

So

$$\|\tilde{u}_{f(x)}(y)\| \le \frac{\|Df(x)^{-1}\|^2}{1 - \|Df(x)^{-1}\|\|u_x(z)\|}\|u_x(z)\|$$

$$\Rightarrow \lim_{y \to f(x)} \|\tilde{u}_{f(x)}(y)\| = 0.$$

GOAL: formulate conditions solely in terms of f

Thm (Inverse function theorem).

Let $f: \mathbb{R}^n \to \mathbb{R}^n$ and $x \in \operatorname{int} \operatorname{Dom}(f)$ be s.t.

$\exists \delta_0 > 0$ be s.t. $B(x, \delta_0) \subseteq \operatorname{Dom}(f)$ and

(1) f differentiable on $B(x, \delta_0)$

(2) Df is continuous at x and $Df(x)^{-1}$ exists.

Then we have: $\exists \delta \in (0, \delta_0)$ s.t.

- f injective and open on $B(x, \delta)$
- f^{-1} differentiable on $f(B(x, \delta))$
- $D(f^{-1})$ is continuous at $f(x)$

Lemma Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $U \subseteq \mathbb{R}^n$ be open and convex be s.t. $\forall x \in U$: $Df(x)$ exists and $M := \sup_{x \in U} \| Df(x) \| < \infty$. Then

$$\forall x, y \in U : \quad \| f(y) - f(x) \| \leq M \, \| y - x \|$$

Pf: Let $x, y \in U$, $\varphi(t) := (1-t)x + ty$. $t \in [0,1]$

Convexity of $U \Rightarrow \forall t \in [0,1]: \varphi(t) \in U$.

Set $h(t) := [f(y) - f(x)] \cdot f(\varphi(t))$

Now: $h'(t) = [f(y) - f(x)] \cdot Df(\varphi(t))(y-x)$

$$\| h'(t) \| \leq \| f(y) - f(x) \| \, M \, \| y - x \|$$

over

(over) $\|f(y)-f(x)\|^2 = h(1)-h(0) \overset{MVT}{=} h'(t)(1-0)$

So $\|f(y)-f(x)\|^2 \leq |h'(t)| \leq M\|f(y)-f(x)\|\,\|y-x\|$

$\Rightarrow \|f(y)-f(x)\| \leq M\|y-x\|$ $\boxtimes$

Pf of IFT: Let $\varepsilon := \frac{1}{2\|Df(x)^{-1}\|}$

and let $\delta\in(0,\delta_0)$ be s.t. $\forall z\in B(x,\delta)$: $\|Df(z)-Df(x)\| < \varepsilon$.

by Lemma from last time with $A = Df(x)$, $B = Df(z)-Df(x)$

we have $\|B\|\,\|A^{-1}\| < 1/2$ and so

$$Df(z)^{-1} \text{ exists} \wedge \|Df(z)^{-1}\| \leq \frac{\|Df(x)^{-1}\|}{1-1/2} = 2\|Df(x)^{-1}\|.$$

$\boxed{\underline{\text{claim}} \quad f \text{ is injective on } B(x,\delta)}$

Pf. Let $y \in f(B(x,\delta))$, define

$$h(z) := z + Df(x)^{-1}(y - f(z))$$

Key fact:

$$y = f(z) \iff h(z) = z$$

Now $Dh(z) = I - Df(x)^{-1} Df(z)$

$$= Df(x)^{-1}(Df(x) - Df(z))$$

So $\|Dh(z)\| \le \|Df(x)^{-1}\| \, \| \text{---} \| \text{---} \|$

$$\le \|Df(x)^{-1}\| \, \varepsilon = \frac{1}{2}$$

by Lemma: $\forall z, \tilde{z} \in B(x,\delta)$:

$$\|h(z) - h(\tilde{z})\| \le \frac{1}{2} \|z - \tilde{z}\|$$

So $h(z) = z \wedge h(\tilde{z}) = \tilde{z} \Rightarrow \|z - \tilde{z}\| \le \frac{1}{2}\|z - \tilde{z}\|$

$$\Rightarrow z = \tilde{z}.$$

So h has at most one fixed point and so f is injective.

$\boxed{\underline{\text{claim}} \quad f \text{ is open on } B(x,\delta)}$

Pf Let $y_0 \in f(B(x,\delta))$. Let $z_0 \in B(x,\delta)$ s.t. $f(z_0) = y_0$

Let $r > 0$ s.t. $\overline{B(z_0, r)} \subseteq B(x,\delta)$

Let $h(z) = z + Df(x)^{-1}(y - f(z))$

Note for $y \in$

$\|h(z) - z_0\|$

Now $\|h(z)$

$\|h(z_0)$

$\Rightarrow \|h(z) - h($

Being a contra

$\Rightarrow$ So map is

$B(x, \delta)$

$)$, define

$x)^{-1}(y - f(z))$

$h(z) = z$

$- Df(x)^{-1} Df(z)$

$Df(x)^{-1}(Df(x) - Df(z))$

$Df(x)^{-1} \| \| - \| - \|$

$Df(x)^{-1} \| \varepsilon = \frac{1}{2}$

by Lemma: $\forall z, \tilde{z} \in B(x, \delta)$:

$$\| h(z) - h(\tilde{z}) \| \leq \frac{1}{2} \| z - \tilde{z} \|$$

So $h(z) = z \wedge h(\tilde{z}) = \tilde{z} \Rightarrow \| z - \tilde{z} \| \leq \frac{1}{2} \| z - \tilde{z} \|$
$\Rightarrow z = \tilde{z}$.

So h has at most one fixed point and so f is injective.

$\boxed{\underline{\text{claim}}\ f \text{ is open on } B(x, \delta)}$

Pf Let $y_0 \in f(B(x, \delta))$. Let $z_0 \in B(x, \delta)$ s.t. $f(z_0) = y_0$

Let $r > 0$ s.t. $\overline{B(z_0, r)} \subseteq B(x, \delta)$

Let $h(z) = z + Df(x)^{-1}(y - f(z))$

Note for $y \in B(y_0, r\varepsilon)$ and $z = f^{-1}(y)$.

$$\| h(z) - z_0 \| \leq \| h(z) - h(z_0) \| + \| h(z_0) - z_0 \|$$
$$\leq \frac{1}{2} r$$

Now $\| h(z) - h(z_0) \| \leq \frac{1}{2} \| z - z_0 \| \leq \frac{1}{2} r$

$$\| h(z_0) - z_0 \| \leq \| Df(x)^{-1} \| \, \| y_0 - y \|$$
$$\leq \| Df(x)^{-1} \| r\varepsilon \leq \frac{1}{2} r$$

$\Rightarrow \| h(z) - h(z_0) \| \leq r \Rightarrow h$ maps $B(z_0, r)$ into itself.

Being a contraction: $\exists$ fixed point.

$\Rightarrow$ So map is open.