Multivariable calculus

GOAL Differential calculus of m-tuples of functions

$$f_1(x_1 \dots x_n), \dots, f_m(x_1 \dots x_n)$$

we regard this as a map $f : \mathbb{R}^n \to \mathbb{R}^m$.

We will think of $\mathbb{R}^n, \mathbb{R}^m$ as metric spaces induced by Euclidean norm $\|x\| := \sqrt{\sum_{i=1}^{n} x_i^2}$ for $x = (x_1 \dots x_n)$

In this sense;

$$\left.\begin{aligned} &z \to x \text{ means } \|z - x\| \to 0 \\ &f(z) \to f(x) \text{ —"— } \|f(z) - f(x)\| \to 0 \end{aligned}\right\} \text{ define notion of } \underline{\text{continuity}}$$

Def Let $f: \mathbb{R}^n \to \mathbb{R}^m$, $x \in \operatorname{int} \operatorname{Dom}(f)$.
We say f is differentiable at x if there exists $m \times n$-matrix A s.t.

$$\lim_{z \to x} \frac{\| f(z) - f(x) - A(z-x) \|}{\| z - x \|} = 0$$

Since A is necessarily unique, denote $Df(x) := A$

We call $Df(x)$ total differential of f at x

Note Every $h: \mathbb{R}^n \to \mathbb{R}^m$ linear map has the form $h(x) = Ax$.

Def Let $f: \mathbb{R}^n \to \mathbb{R}^m$, $x \in \operatorname{int} \operatorname{Dom}(f)$.
Then $\forall i = 1, \dots, m$, $\forall j = 1, \dots, n$:

$$\frac{\partial f_i}{\partial x_j}(x) := \lim_{h \to 0} \frac{f_i(x + h e_j) - f_i(x)}{h}$$

is the partial derivative of f_i w.r.t. x_j.

Similarly, for all $v \in \mathbb{R}^n$:

$$\frac{\partial f_i}{\partial v}(x) = \lim_{h \to 0^+} \frac{f_i(x + hv) - f_i(x)}{h}$$

is the directional derivative of f_i in direction v.

clearly: $\frac{\partial f_i}{\partial x_j}$ …

WATCH: partial…

$f(x,y) := \{ \sqrt{\dots}$ …

draws no relation…

$f(x,y) = \sqrt{x^2 \dots}$

partials don't e…

Lemma If $f: \mathbb{R}^n$ … the partials and …

$\frac{\partial f_i}{\partial v}(x) =$ …

Pf: Set $z = x + h$ …

$\mathbb{R}^n \to \mathbb{R}^m$, $x \in \operatorname{int} \operatorname{Dom}(f)$.

differentiable at x if

$m \times n$-matrix A s.t.

$$\frac{\|f(z) - f(x) - A(z-x)\|}{\|z - x\|} = 0$$

necessarily unique, denote $Df(x) := A$

total differential of f at x

$h: \mathbb{R}^n \to \mathbb{R}^m$ linear map has $h(x) = Ax$.

Def Let $f: \mathbb{R}^n \to \mathbb{R}^m$, $x \in \operatorname{int} \operatorname{Dom}(f)$.

Then $\forall i = 1 \dots m$, $\forall j = 1, \dots, n$:

$$\frac{\partial f_i}{\partial x_j}(x) := \lim_{h \to 0} \frac{f_i(x + h e_j) - f_i(x)}{h}$$

is the partial derivative of f_i w.r.t. x_j.

Similarly, for all $v \in \mathbb{R}^n$:

$$\frac{\partial f_i}{\partial v}(x) = \lim_{h \to 0^+} \frac{f_i(x + hv) - f_i(x)}{h}$$

is the directional derivative of f_i in direction v.

clearly: $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_i}{\partial e_j}$

WATCH: partials may exist yet directionals don't.

$$f(x,y) := \begin{cases} x + y & x = 0 \vee y = 0 \\ \sqrt{x^2 + y^2} & \text{else} \end{cases}$$

shows no relation between the two.

$$f(x,y) = \sqrt{x^2 + y^2}$$

partials don't exist, directionals do.

Lemma If $f: \mathbb{R}^n \to \mathbb{R}^m$ is diff. at $x \in \operatorname{int} \operatorname{Dom}(f)$ then partials and directionals exist. In fact,

$$\frac{\partial f_i}{\partial v}(x) = e_i \cdot Df(x) v$$

Pf: Set $z = x + hv$ in def of total differential:

Corollary If $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \operatorname{int} \operatorname{Dom}(f)$

then
$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}$$

Jacobian matrix

If $f: \mathbb{R}^n \to \mathbb{R}$, we call the vector
$$\nabla f(x) := \left(\frac{\partial f}{\partial x_1}(x) \cdots \frac{\partial f}{\partial x_n}(x) \right)$$
the gradient of f.

Existence of partials NOT sufficient for differentiability

Ex $$f(x) = \begin{cases} \dfrac{x^3}{x^2+y^2} & x^2+y^2>0 \\ 0 & x^2+y^2=0 \end{cases}$$

Not differentiable because $\dfrac{\partial f}{\partial v}(0,0) = \dfrac{v_1^3}{v_1^2+v_2^2}$ NOT linear!

Linearity of directional derivative NOT sufficient either

Ex

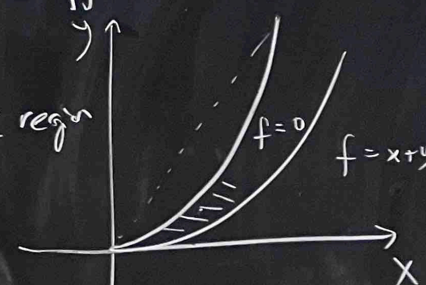
$$f(x,y) := \begin{cases} 0 & x>0 \wedge x^2<y<2x^2 \\ x+y & \end{cases}$$

Then $\dfrac{\partial f}{\partial v}(0,0) = v_1+v_2$ linear

yet: for z in parabolic region

$$\begin{aligned} f(z)-f(0)-(1,1)\cdot(z-0) &= -(1,1)\cdot z \\ &= -(z_1+z_2) \end{aligned}$$

So $\dfrac{\|f(z)-f(0)-(1,1)(z-0)\|}{\|z-0\|} = \dfrac{\|z\|_1}{\|z\|_2} \geq \dfrac{1}{\sqrt{2}}$



Lemma (Sufficient condition)

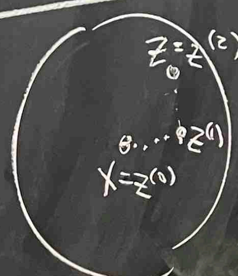
Let $f: \mathbb{R}^n \to \mathbb{R}^m$ and $x \in \operatorname{int} \operatorname{Dom}(f)$.

Assume $\exists \delta > 0$ s.t.

- $B(x, \delta) \subseteq \operatorname{Dom}(f)$
- $\forall i, j$: $\dfrac{\partial f_i}{\partial x_j}$ exists in $B(x, \delta)$
- $\forall i, j$: $\dfrac{\partial f_i}{\partial x_j}$ is continuous at x.

Then f is differentiable at x.

Pf Idea



Given $\varepsilon > 0$

Let $\delta' \in (0, \delta)$ be s.t.

$$\forall i, j: \forall y \in B(x, \delta'): \left| \frac{\partial f_i}{\partial x_j}(y) - \frac{\partial f_i}{\partial x_j}(x) \right| < \varepsilon.$$

Now pick $z \in B(x, \delta')$ and define

$z = (z_1, \ldots, z_n)$
$x = (x_1, \ldots, x_n)$

$$z^{(k)} = x + \sum_{j=1}^{k} (z_j - x_j) e_j$$

Then $x = z^{(0)} \wedge z = z^{(n)}$

So

$$f_i(z) - f_i(x) = \sum_{k=1}^{n} \left[f_i(z^{(k)}) - f_i(z^{(k-1)}) \right]$$

MVT: $\exists y^{(k)}$ on segment $[z^{(k-1)}, z^{(k)}]$

s.t.

$$f_i(z^{(k)}) - f_i(z^{(k-1)}) = \frac{\partial f_i}{\partial x_k}(y^{(k)})(z_k - x_k)$$

This i

$f_i(z)$

So $| f_i$

Hence $\|$

$\mathrm{Dom}(f)$.

$B(x,\delta)$

...s at x.

at x,

Given $\varepsilon > 0$
Let $\delta' \in (0,\delta)$ be s.t.

$\forall i,j$: $\forall y \in B(x,\delta')$: $\left| \frac{\partial f_i}{\partial x_j}(y) - \frac{\partial f_i}{\partial x_j}(x) \right| < \varepsilon$.

Now pick $z \in B(x,\delta')$ and define
$z = (z_1 \dots z_n)$
$x = (x_1 \dots x_n)$

$$z^{(k)} = x + \sum_{j=1}^{k} (z_j - x_j) e_j$$

Then $x = z^{(0)} \wedge z = z^{(n)}$

So
$$f_i(z) - f_i(x) = \sum_{k=1}^{n} \left[f_i(z^{(k)}) - f_i(z^{(k-1)}) \right]$$

MVT: $\exists y^{(k)}$ on segment $[z^{(k-1)}, z^{(k)}]$
s.t.
$$f_i(z^{(k)}) - f_i(z^{(k-1)}) = \frac{\partial f_i}{\partial x_k}(y^{(k)})(z_k - x_k)$$

This implies

$$f_i(z) - f_i(x) - \sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k}(x)(z_k - x_k)$$
$$= \sum_{k=1}^{n} \left(\frac{\partial f_i}{\partial x_k}(y^{(k)}) - \frac{\partial f_i}{\partial x_k}(x) \right)(z_k - x_k)$$

So
$$\left| f_i(z) - f_i(x) - \sum_{k=1}^{n} \frac{\partial f_i}{\partial x_k}(x)(z_k - x_k) \right|$$
$$\leq \varepsilon \sum_{k=1}^{n} |z_k - x_k| \leq \varepsilon \sqrt{n} \, \| z - x \|$$

Hence
$$\| f(z) - f(x) - A(z - x) \| \leq \varepsilon \sqrt{nm} \, \| z - x \|$$

A ← Jacobian matrix.

$\boxtimes$

In order to prove further properties we need to control the size of matrices.

Def Given $m \times n$-matrix A, set

$$\|A\| := \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Ax\|}{\|x\|}$$

Then:

Lemma $\forall A, B$: $m \times n$ matrices $\forall \alpha, \beta \in \mathbb{R}$:

$$\|\alpha A + \beta B\| \leq |\alpha| \, \|A\| + |\beta| \, \|B\|$$

$$\|\alpha A\| = |\alpha| \, \|A\|$$

and $\|A\| = 0 \Rightarrow A = 0$.