

Recall the following notions:

- A **metric space** is a set  $X$  with a **metric**  $\rho: X \times X \rightarrow [0, \infty)$
- An **open ball** of radius  $r$  centered at  $x$  is the set

$$B(x, r) := \{y \in X: \rho(x, y) < r\}$$

- A set  $O \subseteq X$  in a metric space  $(X, \rho)$  is **open** if

$$\forall x \in O \exists r > 0: B(x, r) \subseteq O$$

A set  $C \subseteq X$  is **closed** if  $O := X \setminus C$  is open

- A **topological space** is a set  $X$  with a **topology**  $\mathcal{T}$  = a set of **open** subsets which is required to contain  $\emptyset$  and  $X$  and be closed under arbitrary unions and finite intersections
- A topology  $\mathcal{T}$  is **metrizable** if it arises from a metric

Metric space notions:

- A sequence  $\{x_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}}$  is **convergent** if there is  $x \in X$  with

$$\forall \epsilon > 0 \exists n_0 \in \mathbb{N} \forall n \geq n_0: \rho(x_n, x) < \epsilon$$

We call  $x$  the **limit** of  $\{x_n\}_{n \in \mathbb{N}}$  with notation  $x = \lim_{n \rightarrow \infty} x_n$  or  $x_n \rightarrow x$

- A sequence  $\{x_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}}$  is **Cauchy** if

$$\forall \epsilon > 0 \exists n_0 \in \mathbb{N} \forall n, m \geq n_0: \rho(x_n, x_m) < \epsilon$$

- A metric space is **complete** if all Cauchy sequences are convergent
- A set  $K \subseteq X$  is **sequentially compact** if every sequence from  $K$  contains a convergent subsequence with a limit in  $K$ , i.e.,

$$\forall \{x_n\}_{n \in \mathbb{N}} \in K^{\mathbb{N}} \exists \{n_k\}_{k \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}} \exists x \in K: n_k \rightarrow \infty \wedge x_{n_k} \rightarrow x$$

Topological notions:

- Given a set  $A \subseteq X$ , its **closure** is the set

$$\bar{A} := \bigcap \{C \subseteq X: \text{closed} \wedge A \subseteq C\}$$

and its **interior** is the set

$$\text{int}(A) := \bigcup \{O \subseteq X: \text{open} \wedge O \subseteq A\}$$

Then  $\text{int}(A)$  is open,  $\bar{A}$  is closed and  $\text{int}(A) \subseteq A \subseteq \bar{A}$

- A (topological) **boundary** of  $A \subseteq X$  is the set

$$\partial A := \bar{A} \setminus \text{int}(A)$$

- A set  $K \subseteq X$  is **compact** if every open cover of  $K$  contains a finite subcover, i.e., if for any set  $I$  and any family  $\{O_\alpha\}_{\alpha \in I}$  of open sets,

$$K \subseteq \bigcup_{\alpha \in I} O_\alpha \Rightarrow \exists F \subseteq I: F \text{ finite} \wedge K \subseteq \bigcup_{\alpha \in F} O_\alpha$$

Equivalent formulation: **Cantor's intersection property**

Connecting the two approaches:

- **Sequential characterization of closedness:** A set  $C \subseteq X$  is closed if and only if every convergent sequence from  $C$  has a limit in  $C$ ; i.e.,

$$\forall \{x_n\}_{n \in \mathbb{N}} \in C^{\mathbb{N}} \forall x \in X: x_n \rightarrow x \Rightarrow x \in C$$

- A set  $K \subseteq X$  in a **complete** metric space  $(X, \rho)$  is sequentially compact if and only if it is **closed** and **totally bounded**
- For any metric space  $(X, \rho)$ ,

$$X \text{ sequentially compact} \Leftrightarrow X \text{ compact}$$

Note: AC is needed in the proof

- Metric spaces  $(X, \rho_X)$  and  $(Y, \rho_Y)$
- A function  $f: X \rightarrow Y$  (with  $\text{Dom}(f)$  not necessarily equal to  $X$ )

### Definition (Continuity at a point)

Let  $x_0 \in \text{Dom}(f)$ . We say that  $f$  is **continuous at**  $x_0$  if

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \in \text{Dom}(f): \rho_X(x, x_0) < \delta \Rightarrow \rho_Y(f(x), f(x_0)) < \epsilon$$

- Equivalent formulation

$$\forall \epsilon > 0 \exists \delta > 0: f(B_X(x_0, \delta)) \subseteq B_Y(f(x_0), \epsilon)$$

- Domain matters: Take  $f: \mathbb{Q} \rightarrow \mathbb{R}$  with

$$f(x) := \begin{cases} 0 & \text{if } x < \sqrt{2} \\ 1 & \text{if } x > \sqrt{2} \end{cases}$$

### Definition (Continuity)

We say that  $f$  is **continuous** if  $\forall x \in \text{Dom}(f): f$  is continuous at  $x$

- **Constants** ( $f(x) := y$ ) are continuous
- **Identity map**  $f(x) := x$  is continuous
- **Distance maps**

$$f(x) := \rho(x, y)$$

or

$$f(x) := \inf\{\rho(x, y) : y \in A\}$$

are continuous

**Theorem**

Let  $f, g: X \rightarrow \mathbb{R}$  be continuous at  $x_0$ . Then so are functions  $f + g$  and  $f \cdot g$ .  
If  $g(x_0) \neq 0$ , then also  $f/g$  is continuous at  $x_0$ .

*Proof for  $f + g$ :* Let  $\epsilon > 0$ . Continuity of  $f$  and  $g$  at  $x_0$  implies

$$\exists \delta' > 0 \forall x \in X: \rho_X(x, x_0) < \delta' \Rightarrow |f(x) - f(x_0)| < \epsilon/2$$

and

$$\exists \delta'' > 0 \forall x \in X: \rho_X(x, x_0) < \delta'' \Rightarrow |g(x) - g(x_0)| < \epsilon/2$$

Now set  $\delta := \min\{\delta', \delta''\}$  and note that for all  $x$  with  $\rho_X(x, x_0) < \delta$ :

$$|(f + g)(x) - (f + g)(x_0)| \leq |f(x) - f(x_0)| + |g(x) - g(x_0)| < \epsilon/2 + \epsilon/2 = \epsilon$$

*Proof for  $f \cdot g$ :* Given  $\epsilon > 0$  find  $\delta > 0$  such that for all  $x$  with  $\rho_X(x, x_0) < \delta$ :

$$|f(x) - f(x_0)| < \min\left\{\epsilon, \frac{1}{2} \frac{\epsilon}{\epsilon + |g(x_0)|}\right\}$$

AND

$$|g(x) - g(x_0)| < \frac{1}{2} \frac{\epsilon}{\epsilon + |f(x_0)|}$$

Then  $|f(x)| \leq |f(x) - f(x_0)| + |f(x_0)| \leq \epsilon + |f(x_0)|$  gives

$$\begin{aligned} |(f \cdot g)(x) - (f \cdot g)(x_0)| &= \left| f(x)(g(x) - g(x_0)) + g(x_0)(f(x) - f(x_0)) \right| \\ &\leq |f(x)||g(x) - g(x_0)| + |g(x_0)||f(x) - f(x_0)| \\ &< \frac{1}{2} \frac{\epsilon(\epsilon + |f(x_0)|)}{\epsilon + |f(x_0)|} + \frac{1}{2} \frac{\epsilon|g(x_0)|}{\epsilon + |g(x_0)|} \leq \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

*Proof for  $f/g$ :* similar, see the notes ...

Thanks to algebraic rules we get:

### Corollary

*All polynomials are continuous on  $\mathbb{R}$ . A rational function  $R(x) := P(x)/Q(x)$  where  $P$  and  $Q$  are polynomials is continuous on  $\text{Dom}(R) := \{x \in \mathbb{R} : Q(x) \neq 0\}$ .*

Another way to produce continuous maps:

Lemma (Composition rule)

Suppose  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  and  $x_0 \in X$  obey:

$$x_0 \in \text{Dom}(f) \wedge f \text{ is continuous at } x_0$$

AND

$$f(x_0) \in \text{Dom}(g) \wedge g \text{ is continuous at } f(x_0)$$

Then  $g \circ f$  is continuous at  $x_0$

For each  $\eta > 0$  find  $\epsilon > 0$  so that

$$g(B_Y(f(x_0), \epsilon)) \subseteq B_Z(g \circ f(x_0), \eta)$$

Next find  $\delta > 0$  so that

$$f(B_X(x_0, \delta)) \subseteq B_Y(f(x_0), \epsilon)$$

Then

$$g \circ f(B_X(x_0, \delta)) \subseteq B_Z(g \circ f(x_0), \eta)$$

proving continuity of  $g \circ f$  at  $x_0$

## Theorem (AC)

Let  $f: X \rightarrow Y$  be a function with  $\text{Dom}(f) = X$ . Then for all  $x \in X$ :

$$f \text{ is continuous at } x \Leftrightarrow \left( \forall \{x_n\}_{n \in \mathbb{N}} \in X^{\mathbb{N}}: x_n \rightarrow x \Rightarrow f(x_n) \rightarrow f(x) \right)$$

*Proof of  $\Rightarrow$ :* If  $\epsilon > 0$  and  $\delta > 0$  are s.t.  $f(B_X(x, \delta)) \subseteq B_Y(f(x), \epsilon)$  then

$$\rho(x_n, x) < \delta \Rightarrow \rho_Y(f(x_n), f(x)) < \epsilon$$

*Proof of  $\Leftarrow$ :* If  $f$  NOT continuous at  $x$ , then  $\exists \epsilon > 0$  s.t.

$$\forall \delta > 0: f(B_X(x, \delta)) \setminus B_Y(f(x), \epsilon) \neq \emptyset$$

Pick (using the AC)  $x_n \in B_X(x, \frac{1}{n+1})$  s.t.  $f(x_n) \notin B_Y(f(x), \epsilon)$ . Then

$$x_n \rightarrow x \text{ yet } f(x_n) \not\rightarrow f(x)$$

### Corollary (AC)

*A function (between two metric spaces) is continuous if and only if it turns convergent sequences into convergent sequences.*

**Theorem**

Let  $f: X \rightarrow Y$  be a function with  $\text{Dom}(f) = X$ . Then the following are equivalent:

- (1)  $f$  is continuous
- (2)  $\forall O \subseteq Y: O \text{ open} \Rightarrow f^{-1}(O) \text{ open}$
- (3)  $\forall C \subseteq Y: C \text{ closed} \Rightarrow f^{-1}(C) \text{ closed}$

*Proof of (1)  $\Rightarrow$  (2):* Note that

$$f(B_X(x, \delta)) \subseteq B_Y(f(x), \epsilon) \Leftrightarrow B_X(x, \delta) \subseteq f^{-1}(B_Y(f(x), \epsilon))$$

So given  $O \subseteq Y$  open, for all  $x \in f^{-1}(O)$  we have  $B_X(x, \delta) \subseteq f^{-1}(O)$  for  $\delta$  small

*Proof of (2)  $\Rightarrow$  (1):* Apply (2) to  $O := B_Y(f(x), \epsilon)$  to get  $f^{-1}(B_Y(f(x), \epsilon))$  open. Since  $x \in f^{-1}(B_Y(f(x), \epsilon))$ , there exists  $\delta > 0$  such that

$$B_X(x, \delta) \subseteq f^{-1}(B_Y(f(x), \epsilon))$$

which means  $f(B_X(x, \delta)) \subseteq B_Y(f(x), \epsilon)$

*Proof of (2)  $\Leftrightarrow$  (3):* Apply the identity  $f^{-1}(Y \setminus O) = X \setminus f^{-1}(O)$

**Definition (Continuity in topological spaces)**

Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{S})$  be topological spaces. A map  $f: X \rightarrow Y$  with  $\text{Dom}(f) = X$  is **continuous** if

$$\forall O \in \mathcal{S}: f^{-1}(O) \in \mathcal{T}$$

Note:  $\{f^{-1}(O): O \in \mathcal{S}\}$  the coarsest topology making  $f$  continuous

## Definition

A function  $f: X \rightarrow Y$  with  $\text{Dom}(f) = X$  is said to be

- **open**, or an **open map**, if

$$\forall O \subseteq X: O \text{ open (in } X) \Rightarrow f(O) \text{ open (in } Y)$$

- **closed**, or a **closed map**, if

$$\forall C \subseteq X: C \text{ closed (in } X) \Rightarrow f(C) \text{ closed (in } Y)$$

- Constant functions are closed but generally not open
- For bijective  $f$  we have

$$f \text{ open} \Leftrightarrow f \text{ closed}$$

- For bijective  $f$  we have

$$f \text{ open} \Leftrightarrow f^{-1} \text{ continuous}$$

**Theorem (Continuous image of a compact set is compact)**

Let  $f: X \rightarrow Y$  obey  $\text{Dom}(f) = X$ . Then

$$X \text{ compact} \wedge f \text{ continuous} \Rightarrow f(X) \text{ compact}$$

*Proof:* Let  $\{O_\alpha\}_{\alpha \in I}$  be open cover of  $f(X)$ . Then  $f$  continuous with  $\text{Dom}(f) = X$  implies

$$\{f^{-1}(O_\alpha)\}_{\alpha \in I} \text{ is an open cover of } X$$

Next

$$X \text{ compact} \Rightarrow \exists F \subseteq I: \text{finite} \wedge \bigcup_{\alpha \in F} f^{-1}(O_\alpha) = X$$

But then  $f(X) \subseteq \bigcup_{\alpha \in F} O_\alpha$  so  $f(X)$  is compact