HW#7: due Wed 5/27/2026

This exercise is largely focused on the Stieltjes integral $\int_a^b f \mathrm{d}g$ defined as the limit of the Riemann-Stieltjes sums $S(f, \mathrm{d}g, \Pi)$ for marked partitions Π of $[a, b]$ as the mesh $\|\Pi\|$ tends to zero. We write

$$\mathrm{RS}(g, [a,b]) = \left\{ f\colon [a,b] \to \mathbb{R} : \int_a^b f \mathrm{d}g \text{ exists} \right\}$$

The definition of $\int_a^b f \mathrm{d}g$ in the textbook goes via upper and lower Darboux sums but that makes it limited to g monotone or, by Jordan decomposition, of bounded variation.

Problem 1: Define $f\colon (0,\infty) \to \mathbb{R}$ by

$$f(x) := \int_1^x \frac{1}{t} \mathrm{d}t$$

Prove the following facts:

(1) f is continuous and strictly increasing on $(0,\infty)$
(2) $\forall x, y \in (0,\infty)\colon f(x \cdot y) = f(x) + f(y)$
(3) f^{-1} exists with $\mathrm{Dom}(f^{-1}) = \mathbb{R}$ and obeys $\forall x, y \in \mathbb{R}\colon f^{-1}(x+y) = f^{-1}(x) \cdot f^{-1}(y)$
(4) f^{-1} is continuous on $\mathbb{R}$ and $\exists a > 1\, \forall x \in \mathbb{R}\colon f^{-1}(x) = a^x$

Note: This shows that f^{-1} is an exponential function and f is a logarithm.

Problem 2: Let $a < b$ be reals and let $f, g\colon [a,b] \to \mathbb{R}$ be functions such that

(1) f is Riemann integrable on $[a,b]$, and
(2) g is continuous on $[a,b]$, differentiable on (a,b) with g' Riemann integrable on $[a,b]$.

Prove that $f \in \mathrm{RS}(g, [a,b])$ and

$$\int_a^b f \mathrm{d}g = \int_a^b f(x) g'(x) \mathrm{d}x$$

Then show that also $g \in \mathrm{RS}(f, [a,b])$ and

$$\int_a^b g \,\mathrm{d}f = f(b)g(b) - f(a)g(a) - \int_a^b f(x) g'(x) \mathrm{d}x$$

Note: The existence and Riemann integrability of g' is crucial here as the integrals $\int_a^b f \mathrm{d}g$ and $\int_a^b g \mathrm{d}f$ may fail to exist when g is only continuous!

Problem 3: Let $a < b$ be reals and $\{\alpha_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ a sequence such that $\sum_{n\in\mathbb{N}} |\alpha_n| < \infty$. Given a sequence $\{x_n\}_{n\in\mathbb{N}} \in (a,b]^{\mathbb{N}}$, define $h\colon [a,b] \to \mathbb{R}$ by

$$h(x) := \sum_{n=0}^{\infty} \alpha_n 1_{[x_n,\infty)}(x)$$

where we recall that $1_A(x)$ equals one if $x \in A$ and zero otherwise. (The series converges for each x by our assumptions on $\{\alpha_n\}_{n\in\mathbb{N}}$.) Prove that for all continuous $f\colon [a,b] \to \mathbb{R}$,

$$f \in \mathrm{RS}(h,[a,b]) \ \wedge \ \int_a^b f\mathrm{d}h = \sum_{n=0}^{\infty} \alpha_n f(x_n)$$

Then do the same assuming only that f is bounded and continuous at x_n, for all $n \in \mathbb{N}$. Note: This shows that the Stieltjes integral includes finite sums and convergent series.

Problem 4: Prove the following Mean-Value Theorems: Let $f,g\colon [a,b] \to \mathbb{R}$ be such that f is continuous and g non-decreasing. Then $f \in \mathrm{RS}(g,[a,b])$ and

$$\exists c \in [a,b]\colon \quad \int_a^b f\mathrm{d}g = f(c)\big[g(b) - g(a)\big]$$

Assuming only that f is Riemann integrable and g is non-decreasing, prove that then

$$\exists c \in [a,b]\colon \quad \int_a^b f(x)g(x)\mathrm{d}x = g(a)\int_a^c f(x)\mathrm{d}x + g(b)\int_c^b f(x)\mathrm{d}x$$

Hint: The first statement relies on the Intermediate Value Theorem. For the second statement, write the Riemann integral on the left as $\int_a^b g\mathrm{d}h$ for a suitable h.

Problem 5: Prove Cousin's Theorem: Let $a < b$ be reals and assume that $\mathcal{I}$ is a collection of non-degenerate closed subintervals of $[a,b]$ with the following property: For each $x \in [a,b]$ there is $\delta > 0$ such that all non-degenerate closed intervals $[c,d]$ satisfying

$$[c,d] \subseteq [a,b] \ \wedge \ x \in [c,d] \ \wedge \ d - c < \delta$$

belong to $\mathcal{I}$. Prove that then there is a partition of $[a,b]$ consisting only of intervals in $\mathcal{I}$, i.e., that there are $a = t_0 < t_1 < \cdots < t_n = b$ satisfying

$$\forall i = 1,\dots,n\colon \ [t_{i-1},t_i] \in \mathcal{I}$$

Note: Cousin's theorem ensures that, in the definition of Henstock-Kurzweil integral, for each gauge function there is at least one marked partition obeying the gauge restriction.
