HW#6: due Fri 5/15/2026

Problem 1: Let $h\colon [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$ and let $F\colon \mathbb{R} \to \mathbb{R}$ be bounded and continuous on $\text{Dom}(F)$ which we assume obeys $\overline{\text{Ran}(h)} \subseteq \text{Dom}(F)$. Prove that $F \circ h$ is Riemann integrable on $[a,b]$. (This implies that if f is Riemann integrable, then so is f^2 and, if $f \geqslant 0$, also $\sqrt{f}$, and similarly for other basic functions.)

Problem 2: Let $f\colon [a,b] \to \mathbb{R}$ be bounded and such that

$$\forall x \in (a,b)\colon \lim_{z \to x} f(z) \text{ exists}$$

(Note that this says nothing about continuity of f.) Prove that f is Riemann integrable.

Problem 3: (RUDIN) EX 2, PAGE 138 (Vanishing integral implies vanishing continuous integrand.) Then give an example of a function $f\colon [0,1] \to [0,1]$ such that

$$\bigl\{x \in [0,1]\colon f \text{ NOT continuous at } x\bigr\} \text{ is uncountable and dense in } [0,1]$$

and yet f is Riemann integrable and $\int_0^1 f(x)\mathrm{d}x = 0$.

Problem 4: Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bounded function (with $\text{Dom}(f) = \mathbb{R}$). Define functions $M_f, m_f\colon \mathbb{R} \to \mathbb{R}$ by

$$M_f(x) := \inf_{\delta>0} \sup_{z\colon |z-x|<\delta} f(z) \quad \text{and} \quad m_f(x) := \sup_{\delta>0} \inf_{z\colon |z-x|<\delta} f(z)$$

and, given $s > 0$, let

$$U_s := \bigl\{x \in \mathbb{R}\colon M_f(x) - m_f(x) < s\bigr\}$$

Prove that the following holds for all $s > 0$:

(1) U_s is open
(2) if f is Riemann integrable on $[a,b]$, then U_s is dense in $[a,b]$.

Now use is to conclude that, if $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable, then

$$\bigl\{x \in [a,b]\colon f \text{ continuous at } x\bigr\} \text{ is dense in } [a,b]$$

Problem 5: Let $a < b$ be reals and $f, g\colon [a,b] \to \mathbb{R}$ bounded functions. Prove that the upper Darboux integral is subadditivive,

$$\overline{\int_a^b}(f+g)(x)\mathrm{d}x \leqslant \overline{\int_a^b} f(x)\mathrm{d}x + \overline{\int_a^b} g(x)\mathrm{d}x$$

and the lower Darboux integral is superadditive,

$$\underline{\int_a^b}(f+g)(x)\mathrm{d}x \geqslant \underline{\int_a^b} f(x)\mathrm{d}x + \underline{\int_a^b} g(x)\mathrm{d}x$$

Use these to prove that, if f and g are both Darboux/Riemann integrable, then so is $f+g$ and the integral is additive. Then give explicit examples of f and g for which the above inequalities are strict.

Problem 6: (RUDIN) EX 8, PAGE 138 (Improper integral and integral test)

Problem 7: For each $x > 1$ define $\log(x) := \int_1^x t^{-1}\mathrm{d}t$. Use ideas from previous exercise to prove that

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \log(n)\right]$$

exists in $\mathbb{R}$ and is non-negative.

Problem 8: (RUDIN) EX 10, PAGE 139 (Hölder's inequality) Solve first for the Riemann integral (instead of the Stieltjes integral). You may want to start by proving that if $f\colon [a,b] \to \mathbb{R}$ is integrable, then

$$f \geqslant 0 \;\Rightarrow\; \int_a^b f(x)\mathrm{d}x \geqslant 0$$

and using this to show the *Cauchy-Schwarz inequality*: For all $f, g\colon [a,b] \to \mathbb{R}$ that are Riemann integrable,

$$\left|\int_a^b f(x)g(x)\mathrm{d}x\right| \leqslant \left(\int_a^b f(x)^2\mathrm{d}x\right)^{1/2}\left(\int_a^b g(x)^2\mathrm{d}x\right)^{1/2}$$
