HW#5: due Fri 5/8/2026

The following 6 problems practice multivariate differential calculus. Most of these questions are of computational nature and could be assigned already in MATH 32A, but there is a bit more as some also require proofs of existence of solutions and their differentiability.

Problem 1: Given a differentiable 1-periodic $\varphi\colon \mathbb{R} \to \mathbb{R}$ satisfying $\forall y \in \mathbb{R}\colon |\varphi'(y)| < 1$, prove that there exists $y\colon \mathbb{R} \to \mathbb{R}$ such that $y = y(x)$ solves the equation

$$x = y + \varphi(y)$$

for all $x \in \mathbb{R}$. Then prove that there exists a 1-periodic function $\psi\colon \mathbb{R} \to \mathbb{R}$ such that

$$\forall x \in \mathbb{R}\colon\ y(x) = x + \psi(x)$$

Write a differential equation for ψ in the form $\psi'(x) = F(x, \psi(x))$ for a suitable function $F\colon \mathbb{R}^2 \to \mathbb{R}$ that is expressed in terms of φ.

Problem 2: A metal rod of fixed length $L > 0$ is attached to the positive part of the x axis at one end and the positive part of the y axis at the other end, but can slide freely along each of them. Let Γ be the set of points in $[0, L] \times [0, L]$ that lie above the rod no matter what position it is in. Prove that this set takes the form

$$\Gamma = \big\{(x, y) \in [0, L] \times [0, L]\colon y > \gamma(x)\big\}$$

where $\gamma\colon [0, L] \to [0, L]$ is a convex function. Determine γ explicitly.

Hint: Assume that in any position, the rod is tangent to the graph of γ at exactly one point. Parametrize this point by the slope of the rod and derive suitable equations.

Note: The curve $x \mapsto (x, \gamma(x))$ is the so called *envelope curve* of the set of linear segments representing the rod in its allowed positions.

Problem 3: Suppose that $x = f(u, v)$ and $y = g(u, v)$ for twice continuously differentiable functions $f, g\colon \mathbb{R}^2 \to \mathbb{R}$. Assuming the map $(u, v) \to (x, y)$ to be invertible with the inverse twice continuously differentiable, compute the first and second partial derivatives of u with respect to x and y.

Problem 4: Let $F\colon \mathbb{R}^3 \to \mathbb{R}$ be an everywhere defined, continuously differentiable function with $0 \in \operatorname{Ran}(F)$ and $F^{-1}(\{0\})$ a singleton. Assume that the partial derivatives of f are non-zero at all points. The Implicit function theorem gives us solutions $x = x(y, z)$, $y = y(x, z)$ and $z = z(x, y)$ to $F(x, y, z) = 0$. Prove that, at the solution points,

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$$

(This runs contrary to the quotient interpretation of the Leibniz notation.) What do we get for the analogous product of n partial derivatives when $F\colon \mathbb{R}^n \to \mathbb{R}$ for $n \geqslant 2$?

Problem 5: Let $f, g, h\colon \mathbb{R}^3 \to \mathbb{R}$ be twice continuously differentiable functions with domain $\mathbb{R}^3$. Assume that for each $x \in \mathbb{R}$, the Jacobian matrix of the map

$$\begin{pmatrix} y \\ z \end{pmatrix} \mapsto \begin{pmatrix} g(x,y,z) \\ h(x,y,z) \end{pmatrix}$$

is invertible at all points. Assume there exists at least one point (x_0, y_0, z_0) at which we have $g(x_0, y_0, z_0) = 0 = h(x_0, y_0, z_0)$. Prove that there exists a continuously differentiable function $u\colon \mathbb{R} \to \mathbb{R}$ solving

$$\begin{aligned} f(x,y,z) &= u(x) \\ g(x,y,z) &= 0 \\ h(x,y,z) &= 0 \end{aligned}$$

at all x in an open interval centered at x_0. Then compute u' and, noting that this shows that u is in fact twice continuously differentiable, also u''.

Problem 6: The two-dimensional Laplace operator Δ acts on twice continously differentiable $f\colon \mathbb{R}^2 \to \mathbb{R}$ as

$$\Delta f(x) := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Consider the change of variables to polar coordinates

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

Express Δf in terms of the derivatives with respect to r and θ only.
