HW#3: due Fri 4/24/2026

Problem 1: Let $f\colon \mathbb{R}\to\mathbb{R}$ be non-decreasing with $\text{Dom}(f)=\mathbb{R}$. Prove that $x\mapsto f(x^+)$ is right continuous while $x\mapsto f(x^-)$ is left continuous with both functions non-decreasing. In addition, prove that

$$\forall x\in\mathbb{R}\colon\;\; f(x^-)\leqslant f(x)\leqslant f(x^+)$$

and

$$\forall x,y\in\mathbb{R}\colon\;\; x<y\;\Rightarrow\; f(x^+)\leqslant f(y^-)$$

Problem 2: Given a function $f\colon\mathbb{R}\to\mathbb{R}$ and $a\in\text{int}(\text{Dom}(f))$, prove:

$$f'(a)\text{ exists }\;\Leftrightarrow\;\exists b\in\mathbb{R}\colon\;\lim_{\delta\to0^+}\frac{1}{\delta}\sup_{x\in(a-\delta,a+\delta)}\bigl|f(x)-f(a)-b(x-a)\bigr|=0$$

Also prove that b satisfying the condition on the right is necessarily unique.

Problem 3: Let $f\colon\mathbb{R}\to\mathbb{R}$ be injective and let $x\in\text{int}(\text{Dom}(f))$ be such that $f(x)\in\text{int}(\text{Ran}(f))$ and such that the inverse function f^{-1} of f (with $\text{Dom}(f^{-1}):=\text{Ran}(f)$) is continuous at the point $f(x)$. Assuming $f'(x)\neq0$, prove that f^{-1} is also differentiable at the point $f(x)$ with

$$(f^{-1})'\bigl(f(x)\bigr)=\frac{1}{f'(x)}$$

Problem 4: (RUDIN) EX 1, PAGE 114 (No non-constant 2-Hölder functions.)

Problem 5: (RUDIN) EX 5, PAGE 114 (Increments vanish if derivative vanishes)

Problem 6: (RUDIN) EX 6, PAGE 114 (Conditions for $x\mapsto f(x)/x$ to be increasing.)

Problem 7: (RUDIN) EX 8, PAGE 114 (Continuous $f'\Rightarrow$ uniform differentiability.)

Problem 8: (RUDIN) EX 11, PAGE 115 (Direct computation of f''.)

Problem 9: (Convex functions are one-sided differentiable) Recall that the right and left derivatives of f at x_0 are defined by the right and left limits (read just the top signs or the bottom signs):

$$\frac{\mathrm{d}f}{\mathrm{d}x^{\pm}}(x_0)=f'^{,\pm}(x_0):=\lim_{x\to x_0^{\pm}}\frac{f(x)-f(x_0)}{x-x_0}$$

Let $a<b$ be reals and let $f\colon[a,b]\to\mathbb{R}$ be convex in the sense that

$$\forall x,y\in[a,b]\,\forall\alpha\in[0,1]\colon f(\alpha x+(1-\alpha)y)\leqslant\alpha f(x)+(1-\alpha)f(y)$$

Prove that

(1) $f'^{,+}(x)$ exists for all $x\in[a,b)$ and $f'^{,-}(x)$ exists for all $x\in(a,b]$.

(We do allow the derivatives at a and b to take $\pm\infty$-values.) Then show the following:

(2) f is continuous on (a,b) but not necessarily on $[a,b]$,

(3) $\forall x \in (a,b)\colon f'^{,-}(x) \leqslant f'^{,+}(x)$,
(4) $\forall x, y \in [a,b]\colon x < y \Rightarrow f'^{,+}(x) \leqslant f'^{,-}(y)$,

Assuming that f is continuous on $[a,b]$, prove also that

(5) the following version of Mean-Value Theorem holds

$$\exists x \in (a,b)\colon\ f'^{,-}(x) \leqslant \frac{f(b)-f(a)}{b-a} \leqslant f'^{,+}(x),$$

(6) $f^{'+}$ is right-continuous on $[a,b)$. In fact,

$$\forall x \in [a,b)\colon f'^{,+}(x) = \lim_{z \to x^+} f'^{,+}(z) = \lim_{z \to x^+} f'^{,-}(z)$$

(Similarly, $f^{'-}$ is left-continuous on $(a,b]$ but this can be done by reflection.)

Now use these to solve the next problem.

Problem 10: (RUDIN) EX 14, PAGE 115 (Convexity under positivity of f''.)
