

## HW2: due Fri 4/17/2026

The purpose of this assignment is to give some practice of continuity and limits of functions.

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**Problem 1:** (RUDIN) PAGE 100, EX 15 (Every continuous open mapping  $\mathbb{R} \rightarrow \mathbb{R}$  is monotonic.)

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**Problem 2:** Let  $X$  and  $Y$  be metric spaces with  $X$  totally bounded. Prove that for each  $f: X \rightarrow Y$ ,

$$f \text{ Cauchy continuous} \Rightarrow f \text{ uniformly continuous}$$

(Cauchy continuity thus replaces continuity in non-complete setting.)

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**Problem 3:** (RUDIN) PAGE 100, EX 14 (Fixed point for continuous  $f: I \rightarrow I$ .)

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**Problem 4:** (RUDIN) PAGE 100, EX 17 Note: The conditions and conclusions are restricted to  $x \in E$  where  $f$  has a simple discontinuity.

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**Problem 5:** (RUDIN) PAGE 100, EX 19 (Continuity from intermediate value property.)

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**Problem 6:** Let  $X$  be a metric or topological space. We say that  $X$  is path connected if for every  $x, y \in X$  there is a continuous function  $f: [0, 1] \rightarrow X$  with  $\text{Dom}(f) = [0, 1]$  such that  $f(0) = x$  and  $f(1) = y$ . Prove that

$$X \text{ path connected} \Rightarrow X \text{ connected}$$

(The converse to this fails.)

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**Problem 7:** Let  $(X, \rho)$  be a metric space and  $f: X \rightarrow \mathbb{R}$  a function with  $\text{Dom}(f) = X$ . Define

$$M_f(x) := \inf_{\delta > 0} \sup_{z \in B(x, \delta)} f(z)$$

and

$$m_f(x) := \sup_{\delta > 0} \inf_{z \in B(x, \delta)} f(z)$$

Prove that

$$\forall x \in X: f \text{ continuous at } x \Leftrightarrow m_f(x) = M_f(x)$$

Note: the functions  $M_f(x)$  and  $m_f(x)$  are similar to  $\limsup_{z \rightarrow x} f(z)$  and  $\liminf_{z \rightarrow x} f(z)$  except that they allow "looking" at the value at  $x$  as well.

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**Problem 8:** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be such that

$$\forall r > 0 \forall \epsilon > 0: \{x \in \mathbb{R}: |x| < r \wedge |f(x)| > \epsilon\} \text{ is finite}$$

Prove that

$$\forall x \in \mathbb{R}: \lim_{z \rightarrow x} f(z) = 0$$

Given a sequence  $\{q_n\}_{n \in \mathbb{N}}$  enumerating  $\mathbb{Q}$ , verify that the function

$$f(x) := \begin{cases} \frac{1}{n+1}, & \text{if } x = q_n \text{ for some } n \in \mathbb{N}, \\ 0, & \text{else,} \end{cases}$$

obeys the above condition. This extends Exercise 18 (page 100) from Rudin.

**Problem 9:** Consider a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  and, given reals  $a < b$ , define its total variation on  $[a, b]$  by

$$V(f, [a, b]) := \sup_{\Pi} \sum_{i=1}^n |f(t_i) - f(t_{i-1})|$$

where  $\Pi$  denotes a partition  $\{t_i\}_{i=0}^n$  of  $[a, b]$  such that  $a = t_0 < t_1 < \dots < t_n = b$ . For any  $f: \mathbb{R} \rightarrow \mathbb{R}$  with  $V(f, [a, b]) < \infty$ , prove:

(1) total variation is additive in the sense that

$$\forall t \in (a, b): V(f, [a, b]) = V(f, [a, t]) + V(f, [t, b])$$

(2) if  $f$  is continuous at  $x \in [a, b]$ , then so is  $v$ , where  $v(t) := V(f, [a, t])$ .

**Problem 10:** Let  $f: [a, b] \rightarrow \mathbb{R}$  and let  $G_f := \{(t, f(t)): t \in [a, b]\}$  be its graph. Show that the curve in  $\mathbb{R}^2$  constituting  $G_f$  is rectifiable (meaning: of finite Euclidean length) if and only if  $V(f, [a, b]) < \infty$ .