

HW#1: due Fri 4/10/2026

The purpose of this assignment is to give some additional practice on topics treated in 131AH; in particular, infima/suprema, limits of sequences and infinite series, metric spaces. We then add three additional problems on continuity which is the topic we will open up 131BH with.

Problem 1: Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of subsets of a set B which we note is partially ordered by the \subseteq relation. In light of this (and the fact that suprema w.r.t. \subseteq are achieved by unions and infima by intersections) we define

$$\limsup_{n \rightarrow \infty} A_n := \inf_{n \geq 0} \sup_{m \geq n} A_m = \bigcap_{n \geq 0} \bigcup_{m \geq n} A_m$$
$$\liminf_{n \rightarrow \infty} A_n := \sup_{n \geq 0} \inf_{m \geq n} A_m = \bigcup_{n \geq 0} \bigcap_{m \geq n} A_m$$

We also say

$$\lim_{n \rightarrow \infty} A_n \text{ exists} := \forall x \in B \exists n_0 \in \mathbb{N}: (\forall n \geq n_0: x \in A_n) \vee (\forall n \geq n_0: x \notin A_n)$$

Prove that

$$\lim_{n \rightarrow \infty} A_n \text{ exists} \Leftrightarrow \limsup_{n \rightarrow \infty} A_n = \liminf_{n \rightarrow \infty} A_n$$

(This shows that the use of *limes superior* and *limes inferior* for characterization/definition of the limit works even in partially ordered spaces provided suprema and infima exist.)

Problem 2: (Fekete lemma) Let $\{a_n\}_{n \geq 1}$ be a sequence of non-negative reals that is *sub-additive* in the sense

$$\forall m, n \geq 1: a_{m+n} \leq a_m + a_n$$

Prove that $\lim_{n \rightarrow \infty} \frac{a_n}{n}$ exists (in \mathbb{R}) by showing that, in fact,

$$\lim_{n \rightarrow \infty} \frac{a_n}{n} = \inf_{n \geq 1} \frac{a_n}{n}$$

Problem 3: Prove that

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} \quad \text{as well as} \quad \sum_{n=3}^{\infty} \frac{1}{n(\log n)(\log \log n)^p}$$

converges when $p > 1$ and diverges when $p \leq 1$. (Do not rely on the integral criterion as we have not yet started rigorous treatment of the Riemann integral.)

Problem 4: Prove the following special case of Riemann's rearrangement theorem: For each sequence $\{a_n\}_{n \in \mathbb{N}}$ of reals such that $\sum_{n=0}^{\infty} a_n$ converges conditionally, there exists a bijection $\phi: \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N a_{\phi(n)} = +\infty$$

with the (improper) limit taken in extended reals.

Problem 5: Let (X, ρ) be a metric space and $A \subseteq X$ a (non-empty) compact set. Then for each $x \in X$ there exists a “closest element of A to x ” in the sense

$$\forall x \in X \exists z \in A: \rho(x, z) = \inf\{\rho(x, y) : y \in A\}$$

Problem 6: Let (X, ρ) be a metric space and $f: X \rightarrow X$ an everywhere-defined isometry — i.e., a map with $\text{Dom}(f) = X$ and $\rho(f(x), f(y)) = \rho(x, y)$ for all $x, y \in X$. Prove

$$X \text{ (sequentially) compact} \Rightarrow \text{Ran}(f) = X$$

Hint: Consider the contrapositive.

Problem 7: A faithful map of the United States lies on the dinner table in a house in Denver, CO. Prove that there is one, and only one, point on the map that lies exactly above the physical point (in US territory) it represents. *Note:* This is a word problem that requires a suitable interpretation in metric space theory. Do not assume that the North on the map points in the actual North direction at the map location.

Problem 8: (RUDIN) PAGE 98, EX 2 (For f continuous, $f(\overline{E}) \subseteq \overline{f(E)}$)

Problem 9: (RUDIN) PAGE 98, EX 4 (Continuous image of a dense set is dense.)

Problem 10: (RUDIN) PAGE 99, EX 6 (Continuity characterized by graph of function.)