$$\lim_{h \to 0} \frac{f(x+2h) + f(x) - 2f(x+h)}{h^2}$$

$$\overset{\text{l'Hopital}}{=} \lim_{h \to 0} \frac{2f'(x+2h) - 2f'(x+h)}{2h} =$$

$$\overset{?}{=} \lim_{h \to 0} \frac{2f''(x+2h) - f''(x+h)}{1}$$

$$= \lim_{h \to 0} \left[2\,\frac{f'(x+2h) - f'(x)}{2h} - \frac{f'(x+h) - f'(x)}{h} \right] = 2f''(x) - f''(x)$$

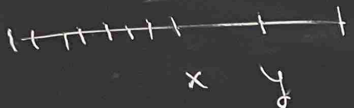
$$= f''(x)$$

Q: $h(x) = V(f, [a,x]) - f(x)$ is $\uparrow$

$a \le x \le y \le b$:

$$V(f,[a,y]) = \sup_{\substack{\Pi \\ \text{partition of } [a,y]}} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|$$

$$\ge \sup_{\substack{\Pi = \Pi' \cup \{y\} \\ \Pi' = \text{partition of } [a,x]}} \text{ —— '' ——}$$



$$= \sup_{\Pi'} \sum_{i=1}^{n-1} |f(t_i) - f(t_{i-1})| + |f(y) - f(x)|$$

$$= V(f,[a,x]) + |f(y) - f(x)|$$

$h(y) - h(x)$

$= V(f,[a,y]) - V(f,[a,x])$

$\quad - [f(y) - f(x)]$

$\ge |f(y) - f(x)| - [f(y) - f(x)] \ge 0.$

Q: Suppose $\forall x \in \mathbb{R}$: $h(x) := \lim_{z \to x^+} f(z)$ exists.
Prove h is RC

$z_n \to x^+$

$\forall \varepsilon > 0 \; \exists z_n' > z_n$: $z_n' - z_n < 2^{-n} \wedge |h(z_n) - f(z_n')| < \varepsilon$

Now $z_n' \to x^+$ so $f(z_n') \to h(x)$

$\Rightarrow h(z_n) \to h(x)$ as well.

Def of limit: $\forall \varepsilon > 0$

This means:

$f((x, x+\delta))$

Pick $z \in (x, x+\delta)$. T

$\Rightarrow \forall z \in (x, x+\delta)$:

$\Rightarrow \lim_{z \to x^+} h(z) = h$

Q: $h(x) = V(f, [a,x]) - f(x)$ is $\uparrow$

$a \le x < y \le b.$

$$V(f,[a,y]) = \sup_{\substack{\Pi \\ \text{partitions of } [a,y]}} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|$$

$$\ge \sup_{\substack{\Pi = \Pi' \cup \{y\} \\ \Pi' = \text{partitions of } [a,x]}} \text{— '' —}$$

$$= \sup_{\Pi'} \sum_{i=1}^{n-1} |f(t_i) - f(t_{i-1})| + |f(y) - f(x)|$$

$$= V(f,[a,x]) + |f(y) - f(x)|$$

$h(y) - h(x)$

$$= V(f,[a,y]) - V(f,[a,x]) - [f(y) - f(x)]$$

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Q: Suppose $\forall x \in \mathbb{R}$: $h(x) := \lim_{z \to x^+} f(z)$ exists. Prove h is RC

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$\Rightarrow h(z_n) \to h(x)$ as well.

Def of limit: $\forall \varepsilon > 0\ \exists \delta > 0\ \forall z \in (x, x+\delta)$: $|h(x) - f(z)| < \varepsilon$.

This means:

$$f((x, x+\delta)) \subseteq (h(x) - \varepsilon,\ h(x) + \varepsilon)$$

Pick $z \in (x, x+\delta)$. Then $h(z) = \lim_{t \to z^+} f(t)$

$$\Rightarrow \forall z \in (x, x+\delta):\ h((x, x+\delta)) \subseteq \overline{f((x, x+\delta))} \subseteq [h(x) - \varepsilon,\ h(x) + \varepsilon]$$

$$\Rightarrow \lim_{z \to x^+} h(z) = h(x)$$

Q: $\mathbb{X}$ compact, $f: X \to X$ s.t. $\left(\begin{matrix}\le \lambda \rho(x,y) \\ (\lambda < 1)\end{matrix}\right)$

$\forall x, y \in \mathbb{X}: x \neq y \Rightarrow \rho(f(x), f(y)) < \rho(x,y)$

Then $\exists x \in X: f(x) = x$.

<u>Note</u> $f: [0,1] \to [0,1]$, $f(x) := x - x^2$

So f may NOT be Banach contraction.

<u>Hint</u> Check $h(x) = \rho(x, f(x))$

<u>claim</u> h continuous

$$h(y) - h(x) = \rho(x, f(x)) - \rho(y, f(y))$$

$$\le \rho(x,y) + \rho(y, f(y)) + \underbrace{\rho(f(y), f(x))}_{\le \rho(x,y)} - \rho(y, f(y))$$

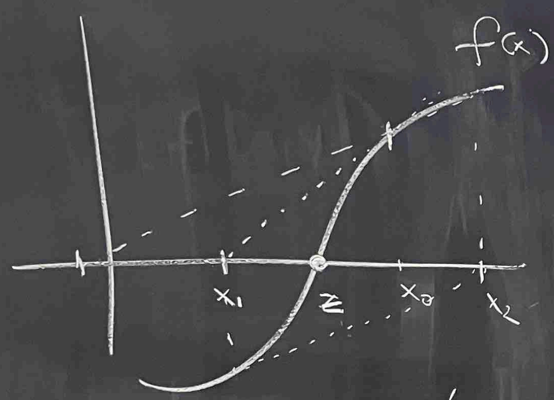
$$|h(y) - h(x)| \le 2\rho(x,y)$$

Now X compact $\Rightarrow$ $\exists\, x \in X : h(x) = \inf_{z \in X} h(z)$.

Assume: $\underline{x \neq f(x)}$. Then $\underbrace{\rho(f(x), f(f(x)))}_{= h(f(x))} < \underbrace{\rho(x, f(x))}_{= h(x) = \inf f}$

So we must have $x = f(x)$.

Newton's method



$$\left.\begin{aligned} y &= f(x_n) + f'(x_n)(x - x_n) \\ y &= 0 \Rightarrow x = x_{n+1} \end{aligned}\right\} \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Q: How fast (if at all) iterates converge?

Assume f twice diff. with

$$B := \sup_{x \in \mathbb{R}} |f''(x)| < \infty$$

$$A := \inf_{x \in \mathbb{R}} f'(x) > 0.$$

Then f has at most one root $\Rightarrow$ call it z.

Then

$$\underset{\substack{z < x \\ \exists t \in (z,x)}}{0 = f(z)} = f(x) + f'(x)(z-x) + \frac{1}{2} f''(t)(z-x)^2$$

Plug $x = x_n$:

$0 = f(x_n) + f'(x_n$

$0 = \underbrace{\frac{f(x_n)}{f'(x_n)} + z}_{= z - x_{n+1}}$

$|x_{n+1} - z| = \Big|\frac{f''(}{2}$

$\leq \frac{B}{2A}$

Check: $|x_n - z| \leq ($

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Q: How fast (if at all) iterates converge?

Assume f twice diff. with

$B := \sup_{x\in\mathbb{R}} |f''(x)| < \infty$

$A := \inf_{x\in\mathbb{R}} f'(x) > 0.$

Then f has at most one root $\Rightarrow$ call it z.

Then

$0 = f(z) = f(x) + f'(x)(z-x) + \frac{1}{2} f''(t)(z-x)^2$

$z<x$

$\exists t \in (z,x)$

Plug $x = x_n$:

$t_n \in (z, x_n)$

$0 = f(x_n) + f'(x_n)(z-x_n) + \frac{1}{2} f''(t_n)(z-x_n)^2$

$0 = \underbrace{\frac{f(x_n)}{f'(x_n)} + z - x_n}_{= z - x_{n+1}} + \frac{1}{2f'(x_n)} f''(t_n)(z-x_n)^2$

$|x_{n+1} - z| = \left|\frac{f''(t_n)}{2f'(x_n)}\right| |z - x_n|^2$

$\leq \frac{B}{2A} |z - x_n|^2$

$\boxed{|z - x_0| \frac{B}{2A} < 1}$

Check: $|x_n - z| \leq \left(\frac{B}{2A}\right)^{2^n - 1} |z - x_0|^{2^n}$