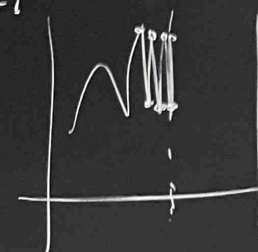
Total variation

$f:[a,b]\to\mathbb{R}$ then $V(f,[a,b]) = \sup\limits_{\Pi} \sum\limits_{i=1}^{n} |f(t_i)-f(t_{i-1})|$

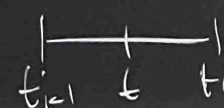


Thm (Jordan) Suppose $V(f,[a,b])<\infty$.

Set $g(x) := V(f,[a,x])$

$h(x) := V(f,[a,x]) - f(x)$

$\boxed{\text{Then } g, h \text{ are nondecreasing and } f = g-h.}$



$$|f(t_i)-f(t_{i-1})| \le |f(t_i)-f(t)| + |f(t)-f(t_{i-1})|$$

__Lemma__ Suppose $\overline{V}(f,[a,b])<\infty$. Then

(1) $\forall x\in(a,b)$: $\overline{V}(f,[a,x])+\overline{V}(f,[x,b]) = \overline{V}(f,[a,b])$

(2) $\forall x\in(a,b]$:

f left cont. at x $\Rightarrow$ $\overline{V}(f,[a,\cdot])$ left cont at x

Similarly for RC.

For (1), insert x into Π defining $\overline{V}(f,[a,b])$ and use

$\sup\{a+b : a\in A \wedge b\in B\} = \sup(A)+\sup(B)$

(2) Let $\varepsilon>0$. Find $\delta>0$ s.t.

$\forall z\in(x-\delta,x]$: $|f(z)-f(x)|<\varepsilon$

By (1): $\overline{V}(f,[a,x])<\infty$

there exists $\Pi=\{t_i\}_{i=0}^{n}$ of $[a,x]$

s.t. $\overline{V}(f,[a,x])\leq\sum_{i=1}^{n}|f(t_i)-f(t_{i-1})|+\varepsilon$

Define $\delta' := \min\{\delta, x-t_{n-1}\}$.

Let $z\in(x-\delta',x]$ and let

$\Pi'=\Pi\cup\{z\}$. Then also

$\overline{V}(f,[a,x])\leq\sum_{i=1}^{n-1}|f(t_i)-f(t_{i-1})|+|f(z)-f(t_{n-1})| + |f(x)-f(z)| + \varepsilon$

$\overline{V}(f,[a,x])\leq \overline{V}(f,[a,z])+|f$

$\leq \overline{V}(f,[a,z])+2\varepsilon$

So $\forall z\in(x-\delta',x]$:

$|\overline{V}(f,[a,z])-\overline{V}(f,$

For RC work with

$\tilde{f}(x)=f(-x)$.

...ppose $\bar V(f,[a,b])<\infty$. Then

...$\in(a,b)$: $\bar V(f,[a,x])+\bar V(f,[x,b]) = \bar V(f,[a,b])$

$x\in(a,b]$:

f left cont. at $x \Rightarrow \bar V(f,[a,\cdot])$ left cont at x

Similarly for RC.

(1), insert x into Π defining $\bar V(f,[a,b])$

...x

$\sup\{a+b : a\in A \wedge b\in B\} = \sup(A)+\sup(B)$

(2) Let $\varepsilon>0$. Find $\delta>0$ s.t.

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By (1): $\bar V(f,[a,x])<\infty$

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s.t. $\bar V(f,[a,x]) \le \sum_{i=1}^n |f(t_i)-f(t_{i-1})| + \varepsilon$

Define $\delta' := \min\{\delta, x-t_{n-1}\}$.

Let $z\in(x-\delta',x]$ and let

$\Pi' = \Pi \cup \{z\}$. Then also

$$\bar V(f,[a,x]) \le \sum_{i=1}^{n-1}|f(t_i)-f(t_{i-1})| + |f(z)-f(t_{n-1})| + |f(x)-f(z)| + \varepsilon$$

$$\bar V(f,[a,x]) \le \bar V(f,[a,z]) + |f(x)-f(z)| + \varepsilon$$

$$\le \bar V(f,[a,z]) + 2\varepsilon$$

So $\forall z\in(x-\delta',x]$:

$$\left|\bar V(f,[a,z]) - \bar V(f,[a,x])\right| \le 2\varepsilon.$$

For RC work with

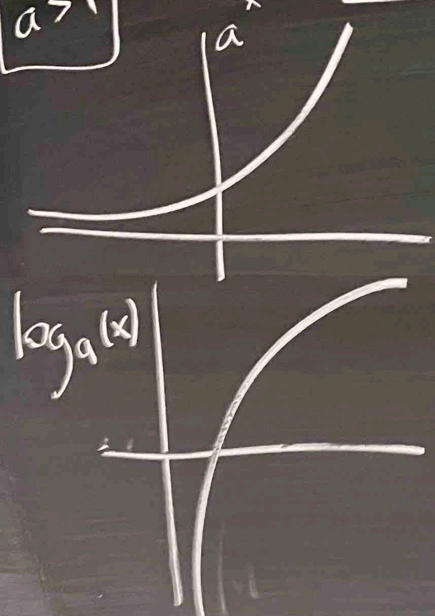
$$\tilde f(x) = f(-x).$$

Derivative of exponentials $(a>0)$

$x \mapsto a^x$ (defined for $x = p/q$ by $\sqrt[q]{a^p}$
extended to $x \in \mathbb{R}$ by unif. continuity)

$f_a(x) := a^x$

$\boxed{a>1}$



know:

- continuous
- strictly monotone if $a \neq 1$
- $\mathrm{Dom}(f_a) = \mathbb{R}$, $\mathrm{Ran}(f_a) \underset{a\neq 1}{=} (0,\infty)$
- admits a continuous inverse
  call it $\log_a$, $\mathrm{Dom}(\log_a) = (0,\infty)$, $\mathrm{Ran}(\log_a) = \mathbb{R}$

$$a^{\log_a(x)} = x, \quad \log_a(a^x) = x$$

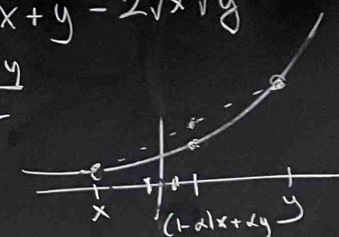
$a \neq 1$

claim $f_a(x) = a^x$ is convex

$$0 \le (\sqrt{x} - \sqrt{y})^2 = x + y - 2\sqrt{x}\sqrt{y}$$

$$\Rightarrow \sqrt{x}\sqrt{y} \le \frac{x+y}{2}$$

Note $a^{\frac{x+y}{2}} = \sqrt{a^x}\sqrt{a^y} \le \frac{1}{2}a^x + \frac{1}{2}a^y$



$$f((1-\alpha)x + \alpha y) \le (1-\alpha)f(x) + \alpha f(y)$$

$\forall x,y \in Dom(f)$ $\forall \alpha \in [0,1]$

Lemma Let $f : \mathbb{R} \to \mathbb{R}$ be s.t.
$\forall x,y \in \mathbb{R}: f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$
Then f continuous $\Rightarrow$ f convex

Pf $f((1-\alpha-\beta)x + (\alpha+\beta)y) = f\left(\frac{(1-\alpha)x + \alpha y}{2} + \frac{(1-\alpha-2\beta) + (\alpha+2\beta)y}{2}\right)$

$$\le \frac{1}{2}f((1-\alpha)x + \alpha y) + \frac{1}{2}f((1-\alpha-2\beta)x + (\alpha+2\beta)y)$$

<u>Claim</u> $\forall n \in \mathbb{N}$ $\forall \alpha = \sum_{i=1}^{n} \sigma_i 2^{-i}$ $\quad \sigma_i \in \{0,1\}$

$\forall x, y \in \mathbb{R}$: $f((1-\alpha)x + \alpha y) \leq (1-\alpha) f(x) + \alpha f(y)$

<u>Pf</u> $n=0$ $\quad \alpha = 0, 1$, trivially TRUE.

Suppose TRUE for n, let

$$\alpha = \sum_{i=1}^{n+1} \sigma_i 2^{-i} \qquad \sigma_{n+1} = 1$$

$$= \tilde{\alpha} + \underbrace{2^{-(n+1)}}_{\beta}$$

$$f((1-\alpha)x + \alpha y) = f((1-\tilde{\alpha}-\beta)x + (\tilde{\alpha}+\beta)y)$$

$$\leq \frac{1}{2} f((1-\tilde{\alpha})x + \tilde{\alpha} y) + \frac{1}{2} f((1-(\tilde{\alpha}+2\beta))x + (\tilde{\alpha}+2\beta)y)$$

$$\overset{\text{ind.}}{\leq} \frac{1}{2}\left((1-\tilde{\alpha}) f(x) + \tilde{\alpha} f(y)\right) + \frac{1}{2}\left((1-\tilde{\alpha}-2\beta) f(x) + (\tilde{\alpha}+2\beta) f(y)\right)$$

$$= (1-\alpha) f(x) + \alpha f(y)$$

Similarly: $f: \mathbb{R}^+ \to \mathbb{R}$

obey $\forall x, y \in \mathbb{R}^+$: $f(xy) = f(x) + f(y)$

Then f cont $\Rightarrow$ f = logarithm

HW3, Pb 9: $f_a(x) = a^x$ is Left/Right differentiable.

$$f_a^{\prime +}(x) = \lim_{h \to 0^+} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h \to 0^+} \frac{a^h - 1}{h} = a^x f_a^{\prime +}(0)$$

Remember:

$\{x \in \mathbb{R}:$

$\Rightarrow \exists x \in \mathbb{R}$

$\Rightarrow$ same ha

We proved:

Then Let $a >$

f_a is differen

$\forall x \in \mathbb{R}$:

$\ldots \sigma_i 2^{-i} \quad \sigma_i \in \{0,1\}$

$\ldots(1-\alpha)x+\alpha y) \le (1-\alpha) f(x) + \alpha f(y)$

trivially TRUE:

for n, let

$\sigma_{n+1} = 1$

$\ldots(n+1)$

$\underbrace{\quad}_{\beta}$

$\ldots 1-\tilde{\alpha}-\beta)x + (\tilde{\alpha}+\beta)y)$

$\ldots \tilde{\alpha}y) + \frac{1}{2} f((1-(\tilde{\alpha}+2\beta))x + (\tilde{\alpha}+2\beta)y)$

$\ldots + \tilde{\alpha} f(y)) + \frac{1}{2}((1-\tilde{\alpha}-2\beta) f(x) + (\tilde{\alpha}+2\beta) f(y))$

$\ldots + \alpha f(y)$

Similar arg: $f : \mathbb{R}^+ \to \mathbb{R}$

obey $\forall x,y \in \mathbb{R}^+ : f(xy) = f(x) + f(y)$

Thm f cont $\Rightarrow$ f = logarithm

HW3, Pb 9: $f_a(x) = a^x$

is Left/Right differentiable.

$$f_a^{\prime,+}(x) = \lim_{h \to 0^+} \frac{a^{x+h} - a^x}{h}$$

$$= a^x \lim_{h \to 0^+} \frac{a^h - 1}{h}$$

$$= a^x f_a^{\prime,+}(0)$$

Remember:

$\{x \in \mathbb{R} : f^{\prime,+}(x) \neq f^{\prime,-}(x)\}$ countable

$\Rightarrow \exists x \in \mathbb{R} : f^{\prime,+}(x) = f^{\prime,-}(x)$

$\Rightarrow$ same happens at all x

We proved:

Thm Let $a > 0$, $f_a(x) = a^x$. Then

f_a is differentiable on $\mathbb{R}$ and

$$\forall x \in \mathbb{R} : f_a'(x) = a^x f_a'(0)$$

Note $b^x = (a^{\log_a(b)})^x = a^{\log_a(b)x}$

$$f_b'(0) = \log_a(b)\, f_a'(0)$$

$a>1$: $b \longmapsto \log_a(b)$ strictly increasing with Range $= \mathbb{R}$.

So $\exists\, b \in \mathbb{R}^+ : f_b'(0) = 1$

Def Euler number e is the unique number s.t $f_e'(0) = 1$.