Topology exercise: X, Y = top. spaces.

Lemma Let $f: X \to Y$ be cont., $\mathrm{Dom}(f) = X$

Thm
- X compact $\Rightarrow$ $f(X)$ compact
- $\forall A \subseteq X: \overline{A} = X \Rightarrow \overline{f(A)} = f(X)$
- X separable $\Rightarrow$ $f(X)$ separable
- X connected $\Rightarrow$ $f(X)$ connected
- X path connected $\Rightarrow$ $f(X)$ path connected

$x, y \in X$
$\varphi: [0,1] \to X$
$\varphi(0) = x,\ \varphi(1) = y$

$u, v \in f(X)$
$x \in f^{-1}(\{u\})$
$y \in f^{-1}(\{v\})$
$f \circ \varphi: [0,1] \to f(X)$

$\mathcal{F}$ = closed set
$\mathcal{G}$ = open set } in top. space X

$\mathcal{F}_\sigma$ = countable unions of elements of $\mathcal{F}$

$\mathcal{G}_\delta$ = —"— intersections of elements of $\mathcal{G}$

claim X metric space. Then

$\mathcal{F} \subseteq \mathcal{G}_\delta$

closed set = cont intersection of open sets

$\mathcal{G} \subseteq \mathcal{F}_\sigma$

open set = countable union of closed sets

$C \in \mathcal{F}$

$$A = \bigcap_{n=0}^{\infty} \bigcup_{x \in C} \underline{B(x, 2^{-n})}$$

$x \in C \Rightarrow x \in A$

$x \notin C \Rightarrow \exists \delta > 0 : B(x,\delta) \cap C = \emptyset$

$\Rightarrow \forall z \in C : x \notin B(z,\delta)$

$\Rightarrow x \notin A$ So $A = C$

Fails in top. spaces (i

$X = \mathbb{N}$

$\mathcal{G} = \{\emptyset, \mathbb{N}\} \cup \{$o

$\mathcal{F} = \{\emptyset, \mathbb{N}\} \cup \{$

... in top. space X

... unions of elements of $\mathcal{F}$

... intersections of elements of $\mathcal{G}$

...ace. Then

$\mathcal{G} \subseteq \mathcal{F}$

open set = countable union of closed sets

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$\Rightarrow \forall z \in C : x \notin B(z,\delta)$

$\Rightarrow x \notin A$

So $A = C$

Fails in top. spaces (in general)

$X = \mathbb{N}$

$\mathcal{G} = \{\emptyset, \mathbb{N}\} \cup \{ \{0, \ldots, n\} : n \in \mathbb{N} \}$

$\mathcal{F} = \{\emptyset, \mathbb{N}\} \cup \{ \{n, n+1, \ldots\} : n \in \mathbb{N} \}$

Power rule

$\forall x, y \in \mathbb{R}: \quad x^n - y^n = (x-y) \sum_{k=0}^{n-1} x^k y^{n-1-k}$

Given $\alpha \in \mathbb{R}$:

Lemma Let $f: (0,\infty) \to \mathbb{R}$ be defined by $f(x) = x^\alpha$. Then $f'(x) = \alpha x^{\alpha - 1}$

$\alpha = -n, \ n \in \mathbb{N}$.

$$\frac{\frac{1}{z} - \frac{1}{x}}{z - x} = \frac{\frac{x-z}{xz}}{z-x} = -\frac{1}{zx}$$

$$\frac{z^{-n} - x^{-n}}{z - x} = \frac{z^{-n} - x^{-n}}{z^{-1} - x^{-1}} \cdot \frac{z^{-1} - x^{-1}}{z - x}$$

$$= -\frac{1}{zx} \sum_{k=0}^{n-1} z^{-k} x^{-(n-1-k)}$$

$$\xrightarrow[z \to x]{} -\frac{1}{x^2} n x^{-n+1} = -n x^{-n-1}$$

$\alpha = \frac{p}{q} \quad p \in \mathbb{Z}, \; q \in \mathbb{N} \setminus \{0\}.$

$$\frac{z^{p/q} - x^{p/q}}{z - x} = \frac{(z^{1/q})^p - (x^{1/q})^p}{z^{1/q} - x^{1/q}} \Bigg/ \frac{(z^{1/q})^q - (x^{1/q})^q}{z^{1/q} - x^{1/q}}$$

$$\xrightarrow[z \to x]{} p\,(x^{1/q})^{p-1} \Big/ q\,(x^{1/q})^{q-1}$$

$$= \frac{p}{q}\, x^{\frac{p}{q} - 1}$$

$$x^{\alpha} = e^{\alpha \log x}$$

$$(x^{\alpha})' = e^{\alpha \log x}\, \alpha \, \frac{1}{x}$$

$$= x^{\alpha - 1}\, \alpha$$

Quotient rule

f, g differentiable at x and $g(x) \neq 0$
then f/g differentiable at x and

$$\left(f/g\right)'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

Pf $\left(f \cdot \frac{1}{g}\right)'(x) = f'(x) \cdot \frac{1}{g(x)} + f(x)\left(-\frac{1}{g(x)^2}\right)g'(x)$

Lemma Let $A \in \mathcal{F}_\sigma$ (on $\mathbb{R}$).
Then $\exists f: \mathbb{R} \to \mathbb{R}$ s.t. $\mathrm{Dom}(f) = \mathbb{R}$
and $\{x \in \mathbb{R} : f \text{ discontinuous at } x\} = A$

Pf Assume first A closed.

$$h(x) = \begin{cases} 0 & x \notin A \\ (1_{\mathbb{Q}}(x) + 1) & x \in A \end{cases}$$

$x \notin$

$x \in$

$x \in \partial A$

$\neq 0$

and

$(x) f(x)$

$g(x)^2) g'(x)$

Lemma Let $A \in \mathcal{F}_\sigma$ (on $\mathbb{R}$).
Then $\exists f: \mathbb{R} \to \mathbb{R}$ s.t. $\mathrm{Dom}(f') = \mathbb{R}$
and $\{x \in \mathbb{R} : f \text{ discontinuous at } x\} = A$

Pf Assume first A closed.

$$h(x) = \begin{cases} 0 & x \notin A \\ (1_{\mathbb{Q}}(x) + 1) & x \in A \end{cases}$$

$x \notin A \Rightarrow \exists \delta > 0 : B(x, \delta) \subseteq A^c$

$h(B(x, \delta)) = \{0\}.$

$x \in \mathrm{int}(A) \Rightarrow \exists \delta > 0 : B(x, \delta) \subseteq A$

$\Rightarrow h = 1 + 1_{\mathbb{Q}}$ on $B(x, \delta)$

$\Rightarrow h(B(x, \delta)) \supseteq \{1, 2\}$

$x \in \partial A \Rightarrow \forall \delta > 0 : h(B(x, \delta)) \supseteq \{0, 1\}$

or $h(B(x, \delta)) \supseteq \{0, 2\}$