**Q1** Two disjoint compact sets have positive distance from each other

$d(x,B) := \inf\{\rho(x,y) : y \in B\}$

$d(A,B) := \inf\{\rho(x,y) : x \in A,\ y \in B\}$

**claim** $\forall K_1, K_2$ seq. compact:

$K_1 \cap K_2 = \phi \Rightarrow d(K_1,K_2) > 0$

$\nexists$ $d(K_1,K_2) = 0 \Rightarrow \exists \{(x_n,y_n)\}_{n\in\mathbb{N}} \in (K_1 \times K_2)^{\mathbb{N}}$

$\rho(x_n,y_n) \to 0 \Rightarrow \exists \{n_k\}: x_{n_k} \to x,\ y_{n_k} \to x$

$\exists x \in K_1 \cap K_2$

$\exists x \in A\ \exists y \in B \quad \ldots \quad \exists (x,y) \in A \times B$

$\{A_n\}_{n\in\mathbb{N}} \quad A_n \neq \phi$

**Axiom of choice:**

$\forall I\ \exists \{A_\alpha\}_{\alpha\in I}:$

$$\left(\forall \alpha \in I : A_\alpha \neq \phi\right) \Rightarrow \bigtimes_{\alpha\in I} A_\alpha \neq \phi$$

$\exists f$ function: $\forall A \neq \phi : f(A) \in A$

**A Modulus of continuity** $\omega : [0,\infty) \to [0,\infty)$

$\uparrow$, $\omega(0) = 0$, $\forall t > 0 : \omega(t) > 0$

$\forall x,y \in X: \ \rho_Y(f(x), f(y)) \leq \omega(\rho_X(x,y))$

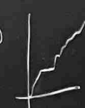
**claim** Every uniformly continuous function admits a modulus of continuity

Given $\varepsilon > 0\ \exists \delta > 0\ \forall x,y \in X: \rho_X(x,y) < \delta \Rightarrow \rho_Y(f(x),f(y)) \leq \varepsilon$

$\delta(\varepsilon) := \sup\{\delta > 0 : \text{———} \| \text{———}\}$

$\varepsilon \mapsto \delta(\varepsilon) \uparrow$, $\delta(0) = 0$

$\omega(r) = \delta^{-1}(r)$



Ex $\omega(r) = A r$ Lipschitz

$\omega(r) = A r^\alpha$ $\alpha$-Hölder

$$\rho(f(x), f(y)) \leq A \rho(x, y)$$

$\underline{\text{Ex}}$ Let $f : X \to Y$, $\mathrm{Dom}(f) = X$, $\mathcal{T}$ topology on $Y$

claim: $\{ f^{-1}(O) : O \in \mathcal{T} \}$ is a topology on $X$.

coarsest topology that makes $f$ continuous

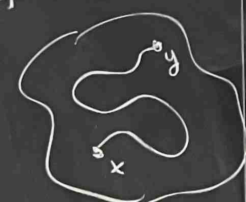
$$f^{-1}\left(\bigcup_{\alpha \in I} A_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(A_\alpha)$$

$$f^{-1}\left(\bigcap_{\alpha \in I} A_\alpha\right) = \bigcap_{\alpha \in I} f^{-1}(A_\alpha)$$

$f$ constant : $\{ f^{-1}(O) : O \in \mathcal{T} \} = \underline{\{ \emptyset, X \}}$

**Ex** $X$ connected if it cannot be partitioned into the union of two nonempty open sets.

open
↓
closed

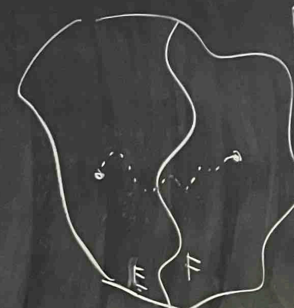


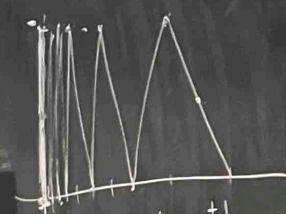

**Def** $X$ is **path connected** if $\forall x, y \in X \ \exists f : [0,1] \to X$ continuous s.t. $f(0) = x \wedge f(1) = y$.

**Thm** (HW2)

$X$ path connected $\Rightarrow$ $X$ connected

$\mathbb{R}^3 \setminus \mathbb{R}$



NOT path connected

Yes connected