45. Analytic functions

We will now move to discuss a class of analytic functions that contain many standard functions in analysis. Analytic functions can be thought of as the analyst's version of the algebraist's concept of a polynomial.

45.1 Definition and characterization.

We have shown that polynomials are dense in $C([a,b],\mathbb{R})$. This leads to the following question: Given a function $f \in C([a,b],\mathbb{R})$ and a sequence of polynomials $\{P_n\}_{n\in\mathbb{N}}$ such that $P_n \to f$, do the coefficients of the polynomials P_n converge? This turns out to be true but only for functions identified in the following definition:

Definition 45.1 (Analytic functions) *Let $f\colon \mathbb{R} \to \mathbb{R}$ (with* $\mathrm{Dom}(f)$ *general) and $x_0 \in \mathbb{R}$. We say that f is* (real) analytic at x_0 *if there exists $\{c_n\}_{n\in\mathbb{N}}$ and $r \in (0,+\infty]$ such that*

$$(x_0 - r, x_0 + r) \subseteq \mathrm{Dom}(f) \tag{45.1}$$

$$\limsup_{n\to\infty} |c_n|^{1/n} \leqslant \frac{1}{r} \tag{45.2}$$

and

$$\forall x \in (x_0 - r, x_0 + r)\colon \quad f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n \tag{45.3}$$

where (45.2) *ensures that the series converges absolutely. We say that f is* analytic on $A \subseteq \mathbb{R}$ *if $\forall x_0 \in A\colon$ f is analytic at x_0.*

The preffix "real" is sometimes added to make distinction between analyticity of real-valued functions and that of complex valued functions. Since, as it turns out, there is actually no significant difference, we will use just the plain adjective "analytic."

Analytic functions are the nicest functions besides polynomials. Indeed, we have:

Lemma 45.2 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be analytic at x_0 with* (45.3) *for some $r > 0$. Then f is arbitrarily continuously differentiable on $(x_0 - r, x_0 + r)$ and*

$$\forall n \in \mathbb{N}\colon \; c_n = \frac{f^{(n)}(x_0)}{n!} \tag{45.4}$$

In particular, the infinite series representation (45.3) *is unique.*

Proof. By Corollary 40.4, the series in (45.3) is arbitrarily differentiable with

$$\forall k \in \mathbb{N}\colon \; f^{(k)}(x) = \sum_{n=k}^{\infty} c_n \left(\prod_{i=0}^{k-1} (n - i) \right) (x - x_0)^{n-k} \tag{45.5}$$

valid for all $x \in (x_0 - r, x_0 + r)$. Setting $x := x_0$ only the term with $n = k$ survives and we get $f^{(k)}(x_0) = c_k k!$. $\square$

The above shows that (45.3) can be cast as

$$\forall x \in (x_0 - r, x_0 + r)\colon \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n. \tag{45.6}$$

The series on the right is then called the *Taylor series* associated with f at x_0. Since this series is expressed entirely using attributes derived from f, this suggests a natural question: Suppose f is arbitrarily continuously differentiable at (and thus in a neighborhood of) x_0 and the associated Taylor series converges on an open interval containing x_0. Is the series equal to f there?

As much as this may sound plausible, the answer to this is in the resolute negative. A standard counterexample is the function

$$f(x) := \begin{cases} \exp\{-1/x^2\}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \tag{45.7}$$

which is arbitrarily continuously differentiable on $\mathbb{R}$ (including at zero) with all derivatives at zero vanishing. This means that the associated Taylor series at $x_0 = 0$ is zero, yet f clearly is not.

The question is then: What property (such as growth estimate etc) of the function and its derivatives ensures analyticity? We answer this in:

Lemma 45.3 *Let $f\colon \mathbb{R} \to \mathbb{R}$ and let $x_0 \in \operatorname{int}(\operatorname{Dom}(f))$. The following are equivalent:*

(1) *f is analytic at x_0,*
(2) *there exists $\epsilon > 0$ such that f is arbitrarily differentiable on $(x_0 - \epsilon, x_0 + \epsilon)$ and*

$$\exists A \in (0, \infty)\, \forall n \geqslant 0\colon \quad \sup_{x \in (x_0 - \epsilon, x_0 + \epsilon)} \big|f^{(n)}(x_0)\big| \leqslant A\epsilon^{-n} n! \tag{45.8}$$

In particular,

Proof. For implication (1)⇒(2), assume the representation (45.3) and let $\tilde r \in (0, r)$. The convergence forces the coefficients to be bounded, which means that

$$A' := \sup_{n \in \mathbb{N}} |c_n| \tilde r^n < \infty \tag{45.9}$$

implying $|c_n| \leqslant A' \tilde r^{-n}$ for all $n \in \mathbb{N}$. For $x \in (x_0 - \tilde r, x_0 + \tilde r)$, formula (45.5) then shows

$$\big|f^{(k)}(x)\big| \leqslant A' \tilde r^{-k} \sum_{n=k}^\infty \left(\prod_{i=0}^{k-1} (n-i) \right) \big(|x - x_0|/\tilde r\big)^{n-k} \tag{45.10}$$

We now interpret the sum on the right-hand side as the k-th derivative of the geometric series $\sum_{n=0}^\infty a^n$ at $a := |x - x_0|/\tilde r$. Indeed, for $a \in (0,1)$ we have

$$\sum_{n=k}^\infty \left(\prod_{i=0}^{k-1} (n-i) \right) a^{n-k} = \frac{\mathrm{d}^k}{\mathrm{d}a^k} \frac{1}{1-a} = \frac{k!}{(1-a)^{k+1}} \tag{45.11}$$

and so

$$\big|f^{(k)}(x)\big| \leqslant \frac{A' \tilde r^{-k} k!}{(1 - \tilde r^{-1}|x - x_0|)^{k+1}} = \frac{A' \tilde r}{(\tilde r - |x - x_0|)^{k+1}} k! \tag{45.12}$$

Taking $\epsilon \in (0, \tilde r/2)$, for $|x - x_0| < \epsilon$ we then have $\tilde r - |x - x_0| > \epsilon$ and (45.12) is at most $3A' \tilde r \epsilon^{-1-k} k!$. This gives (45.8) with $A := 3A' \tilde r \epsilon^{-1}$.

For (2)⇒(1) we note that, by Taylor's theorem, for all $x \in (x_0-\epsilon, x_0+\epsilon)$ and all $n \in \mathbb{N}$ there exists $\xi \in (x_0-\epsilon, x_0+\epsilon)$ — in fact, with ξ between x and x_0 — such that

$$\left|f(x)-\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\right| \leqslant \frac{f^{(n+1)}(\xi)}{(n+1)!}|x-x_0|^{n+1} \tag{45.13}$$

Under the condition (45.8), the right-hand side is bounded by $A(|x-x_0|/\epsilon)^{n+1}$ which tends to zero as $n \to \infty$. The Taylor series associated with f converges and equals f on $(x_0-\epsilon, x_0+\epsilon)$ proving that x_0 is analytic there. □

The key to analyticity is that, not only are the derivatives of f well behaved at x_0 to allow for the Taylor series to converge, but they are similarly well-behaved in a whole neighborhood of x_0. The above argument actually gives us a bit more:

Corollary 45.4 *Suppose that f admits the power series representation* (45.3) *that converges at all $x \in (x_0-r, x_0+r)$. Then f is analytic on (x_0-r, x_0+r).*

Proof. Let $x \in (x_0-r, x_0+r)$ and $\tilde r \in (|x-x_0|, r)$. Let $\epsilon > 0$ be such that $2\epsilon < \tilde r - |x-x_0|$. Then for all $\tilde x \in (x-\epsilon, x+\epsilon)$, which is a subset of $(x_0-\tilde r, x_0+\tilde r)$, the bound (45.12) along with the fact that $\tilde r - |\tilde x - x_0| \geqslant \tilde r - |x-x_0| - \epsilon \geqslant \epsilon$ shows $|f^{(k)}(\tilde x)| \leqslant A'\tilde r\epsilon^{-(k+1)}k!$. This gives (45.8) with $A := A'\tilde r\epsilon^{-1}$ thus proving that f is analytic at x. □

As a consequence of the above observations we get:

Corollary 45.5 *Let $f\colon \mathbb{R} \to \mathbb{R}$. The set $A := \{x \in \mathrm{Dom}(f)\colon f \text{ analytic at } x\}$ is open. All derivatives and antiderivatives (a.k.a. primitives) of f are analytic on A.*

45.2 Examples.

Let us move to discuss the basic examples of analytic functions. The immediate examples are the polynomials whose power series representation are the polynomials themselves. We have also seen the functions exp, sin and cos that are defined by an everywhere convergent power series (centered at zero) which by Corollary 45.4 means that they are analytic on all of $\mathbb{R}$. (In complex analysis, they are analytic on all of $\mathbb{C}$ and are thus called *entire*.)

Another standard example of a function that is not analytic everywhere is the function $f(x) := \frac{1}{1-x}$ with domain $\mathrm{Dom}(f) := \mathbb{R} \smallsetminus \{1\}$. The geometric series representation

$$\forall x \in (-1,1)\colon\ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{45.14}$$

shows that f is analytic on $(-1,1)$. But the function is actually analytic on its entire domain as is seen by taking $x_0 \in \mathbb{R} \smallsetminus \{1\}$ and writing

$$\frac{1}{1-x} = \frac{1}{1-x_0}\frac{1}{1-\frac{x-x_0}{1-x_0}} \overset{|\frac{x-x_0}{1-x_0}|<1}{=} \sum_{n=0}^{\infty}\frac{1}{(1-x_0)^{n+1}}(x-x_0)^n \tag{45.15}$$

The condition $|\frac{x-x_0}{1-x_0}| < 1$ translates into $|x-x_0| < |1-x_0|$ which means that we just need x to be closer to x_0 than x_0 is to 1; i.e., $x \in (x_0-|1-x_0|, x_0+|1-x_0|)$. The power series converges on a largest interval centered at x_0 that fits into $\mathbb{R} \smallsetminus \{1\}$.

A simple variant on this is the function $f(x) := \frac{1}{1+x^2}$ with $\mathrm{Dom}(f) := \mathbb{R}$ for which we get the series representation

$$\forall x \in (-1,1)\colon\ \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \tag{45.16}$$

This shows that f is analytic on $(-1,1)$. The same kind of rewrite as above (albeit more complicated due to the square getting in the way) shows that f is actually analytic on all of $\mathbb{R}$. Note, however, that the radius of convergence of the resulting power series is always finite (e.g., equal to one in (45.16)). This is because the function has singularities in the complex plane — the denominator vanishes at $x = \pm\mathrm{i}$ — which are "felt" by the power series even on the real line.

In homework, we introduce the function

$$\log(x) := \int_1^x \frac{1}{t}\mathrm{d}t \tag{45.17}$$

with $\mathrm{Dom}(\log) = (0,\infty)$ and proved that

$$\forall x,y \in (0,\infty)\colon\ \log(x\cdot y) = \log(x) + \log(y) \tag{45.18}$$

and that log is the inverse of exp. It will not surprise us that:

Lemma 45.6 log *is analytic on its domain, i.e., on* $(0,\infty)$.

Proof. We will first prove that log is analytic at $x_0 = 1$. For this we use the substitution $t \mapsto 1 - t$ to rewrite the defining expression (45.17) as

$$\log(x) = -\int_0^{1-x} \frac{1}{1-t}\mathrm{d}t \tag{45.19}$$

For $|1-x| < 1$ we may expand the integrand into an infinite power series in t and then swap the sum with the integral (thanks to uniform convergence) to get

$$\log(x) = -\sum_{n=0}^{\infty} \int_0^{1-x} t^n \mathrm{d}t = -\sum_{n=0}^{\infty} \frac{(1-x)^{n+1}}{n+1} = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}(x-1)^m \tag{45.20}$$

For general $x_0 \in (0,\infty)$ we then use (45.18) to write assuming $x/x_0 \in (0,2)$ that

$$\begin{aligned} \log(x) &= \log(x_0) + \log(x/x_0) \\ &= \log(x_0) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}(x/x_0 - 1)^m \\ &= \log(x_0) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}x_0^{-m}}{m}(x - x_0)^m \end{aligned} \tag{45.21}$$

Hence, log is analytic on all of $(0,\infty)$. □

45.3 Operations on analytic functions.

We will now proceed to discuss operations that preserve (perhaps under natural conditions) analyticity. We start by the algebraic operations:

Lemma 45.7 *Suppose f and g are analytic on an open set $A \subseteq \mathbb{R}$. Then*

(1) *$f + g$ and $f \cdot g$ are analytic on A,*
(2) *if $\forall x \in A\colon g(x) \neq 0$ then also f/g is analytic on A.*

Proof of (1). Pick $x_0 \in A$ and assume that, for some $\epsilon > 0$ and all $x \in (x_0 - \epsilon, x_0 + \epsilon)$,

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \ \wedge\ g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n \tag{45.22}$$

with the series absolutely convergent. Mertens' Theorem for the Cauchy product then tells us that

$$f \cdot g(x) = \sum_{n=0}^{\infty} \Bigl(\sum_{k=0}^{n} a_n b_{n-k}\Bigr)(x - x_0)^n \tag{45.23}$$

with the series converging absolutely. □

For the second part, we first prove a more general fact:

Lemma 45.8 *Let $g\colon \mathbb{R} \to \mathbb{R}$ be analytic at x_0 and $f\colon \mathbb{R} \to \mathbb{R}$ analytic at $g(x_0)$. Then $f \circ g$ is analytic at x_0. In fact, there is $\delta > 0$ such that $f \circ g$ admits a power series representation*

$$f \circ g(x) = \sum_{\ell=1}^{\infty} \left(\sum_{n=0}^{\ell} \sum_{\substack{k_1,\dots,k_n \geqslant 1 \\ k_1 + \dots + k_n = \ell}} \Bigl(\prod_{i=1}^{n} b_{k_i} \Bigr) \right) (x - x_0)^{\ell} \tag{45.24}$$

that converges absolutely for all $x \in (x_0 - \delta, x_0 + \delta)$. (The sums in the large partentheses contain only finitely many terms.)

Proof. Suppose f and g admit the representation

$$f(y) = \sum_{n=0}^{\infty} a_n \bigl(y - g(x_0)\bigr)^n \ \wedge\ g(x) = \sum_{k=0}^{\infty} b_k (x - x_0)^k \tag{45.25}$$

where the series are absolutely convergent whenever $|x - x_0| < r_0$ and $|y - g(x_0)| < \tilde{r}_0$. The fact that the power series converges means that, given $r \in (0, r_0)$ and $\tilde{r} \in (0, \tilde{r}_0)$, there is $A \in [1, \infty)$ such that

$$\forall n \in \mathbb{N}\colon |a_n| \leqslant A r^{-n} \ \wedge\ |b_n| \leqslant A \tilde{r}^{-n} \tag{45.26}$$

Moreover, given $\epsilon \in (0, r)$, there is $N_0 \in \mathbb{N}$ such that forall $N \geqslant N_0$,

$$\left| f(y) - \sum_{n=0}^{N} a_n \bigl(y - g(x_0)\bigr)^n \right| < \epsilon \ \wedge\ \left| g(x) - g(x_0) - \sum_{k=1}^{N} b_k (x - x_0)^k \right| < \epsilon/2 \tag{45.27}$$

whenever $|x - x_0| < r$ and $|y - g(x_0)| < \tilde{r}$. Here we used that $b_0 = g(x_0)$.

Using the fact that $|g(x) - g(x_0)|$ is continuous, we can find $\delta \in (0, r)$ such that

$$\forall x \in (x_0 - \delta, x_0 + \delta)\colon\ \bigl|g(x) - g(x_0)\bigr| < \epsilon/2 \ \wedge\ \left|\sum_{n=0}^{N} b_n (x - x_0)^n\right| < \epsilon \tag{45.28}$$

where the second bound follows from the first using the second inequality in (45.27). The formula $a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-k}$ then tells us

$$\sup_{x\in(x_0-\delta,x_0+\delta)}\left|\bigl(g(x)-g(x_0)\bigr)^n - \left(\sum_{k=1}^N b_k(x-x_0)^k\right)^n\right| \leqslant \epsilon\epsilon^{n-1}n = n\epsilon^n \tag{45.29}$$

whenever $|x-x_0|<\delta$. Plugging (45.29) in the first series in (45.27) then shows

$$\left| f\circ g(x) - \sum_{n=0}^N a_n\left(\sum_{k=1}^N b_k(x-x_0)^k\right)^n\right| \leqslant \sum_{n=1}^N n|a_n|\epsilon^n \tag{45.30}$$

whenever $|x-x_0|<\delta$. Here the sum on the right starts from $n=1$ because the quantity on the left of (45.29) vanishes when $n=0$ and so no error arises from there.

The error on the right of (45.30) is bounded readily via (45.26) as:

$$\sum_{n=1}^N n|a_n|\epsilon^n \leqslant A\sum_{n=1}^N n(\epsilon/r)^n \leqslant A\frac{\epsilon/r}{(1-\epsilon/r)^2} \tag{45.31}$$

where we used that $\epsilon/r<1$ and invoked (45.11) for $k:=1$. For the N-dependent term inside the absolute value in (45.30) we use the distributive law to get

$$\begin{aligned}\sum_{n=0}^N a_n\left(\sum_{k=1}^N b_k(x-x_0)^k\right)^n &= \sum_{n=0}^N a_n \sum_{1\leqslant k_1,\dots,k_n\leqslant N}\left(\prod_{i=1}^n b_{k_i}\right)(x-x_0)^{k_1+\dots+k_n}\\ &= \sum_{\ell=1}^{N^2}\left(\sum_{n=0}^{\min\{\ell,N\}}\sum_{\substack{1\leqslant k_1,\dots,k_n\leqslant N\\ k_1+\dots+k_n=\ell}}\left(\prod_{i=1}^n b_{k_i}\right)\right)(x-x_0)^\ell\end{aligned} \tag{45.32}$$

where we curbed the sum over n using the fact that, since all k_i's are at least one, we must have $n\leqslant\ell$ under the sums.

Using the bounds (45.26) the product of the b_{k_i}'s is at most $A^n\tilde r^{-\ell}$ while the sum over k_i's has at most $\binom{\ell}{n}$ terms (by a combinatorial argument of distributing ℓ balls into n bins so that each bin receives at least one ball) we have

$$\left|\sum_{n=0}^{\min\{\ell,N\}}\sum_{\substack{1\leqslant k_1,\dots,k_n\leqslant N\\ k_1+\dots+k_n=\ell}}\left(\prod_{i=1}^n b_{k_i}\right)\right| \leqslant \sum_{n=0}^{\ell} A^n\binom{\ell}{n}\tilde r^{-\ell} = \bigl[(A+1)/\tilde r\bigr]^\ell \tag{45.33}$$

where we used the Binomial Theorem in the last step. (This is regardless of N.) Assuming δ was chosen so small that

$$\delta(A+1)/\tilde r < 1 \tag{45.34}$$

the infinite series

$$\sum_{\ell=1}^{\infty}\left(\sum_{n=0}^{\ell}\sum_{\substack{k_1,\dots,k_n\geqslant 1\\ k_1+\dots+k_n=\ell}}\left(\prod_{i=1}^n b_{k_i}\right)\right)(x-x_0)^\ell \tag{45.35}$$

converges uniformly in $x \in (x_0 - \delta, x_0 + \delta)$. Moreover, since the restriction to $\ell \leqslant N$ on the right of (45.32) effectively nulls the restrictions on N, in this range of x we also get

$$\begin{aligned}\Biggl|\sum_{n=0}^{N} a_n \Bigl(\sum_{k=1}^{N} b_k (x-x_0)^k\Bigr)^n \\ - \sum_{\ell=1}^{\infty}\Biggl(\sum_{n=0}^{\ell} \sum_{\substack{k_1,\dots,k_n \geqslant 1\\ k_1+\dots+k_n=\ell}} \Bigl(\prod_{i=1}^{n} b_{k_i}\Bigr)\Biggr)(x-x_0)^\ell\Biggr| \leqslant 2\sum_{\ell=N+1}^{\infty}\bigl[\delta(A+1)/\tilde r\bigr]^\ell\end{aligned} \tag{45.36}$$

where the factor 2 arises from using the bound (45.33) once for $\ell > N$ in the sum in (45.32) and the second time to the corresponding terms in the series in (45.35).

The right-hand side of (45.36) is less than $\epsilon > 0$ once N is sufficiently large. We have thus proved that $f \circ g(x)$ lies within

$$\epsilon + A\frac{\epsilon/r}{(1-\epsilon/r)^2} \tag{45.37}$$

of the series in (45.35) uniformly in $x \in (x_0 - \delta, x_0 + \delta)$. But $\epsilon > 0$ was arbitrary (sufficiently small), so $f \circ g(x)$ equals the series on this interval. □

Proof of (2) in Lemma 45.7. Since $h(z) := 1/z$ is analytic away from $z = 0$ (see the discussion after (45.14)) and $g(x_0) \neq 0$, the composition $h \circ g = 1/g$ is analytic at x_0. By part (1), so is then f/g. □

Yet more complicated arguments give:

Lemma 45.9 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be analytic at x_0 with $f'(x_0) \neq 0$. Then f^{-1} exists on a neighborhood of $f(x_0)$ and is analytic there.*

The idea of the proof is to use the inversion formula $(f^{-1})'(x) = 1/f'(f(x))$ to progressively control all derivatives. Formulas for this exist (and are referred to as the Langrange Inversion Theorem) but the calculations are quite complicated. (Details have in fact been supplied much later by Bürrman.)

The reason why we do not go into these any further is that all of the above proofs can be done far more elegantly using complex analysis. Indeed, there one shows that a function $f\colon \mathbb{C} \to \mathbb{C}$ defined on a neighborhood of $z_0 \in \operatorname{int}(\operatorname{Dom}(f))$ obeys

$$f \text{ analytic at } z_0 \;\Leftrightarrow\; \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} \text{ exists} \tag{45.38}$$

The property on the right, referred to as f being *holomorphic at* z_0, is simply the existence of a complex derivative. Note, however, that this is not the same as the derivative of a real-valued functions. Indeed, z can approach z_0 from all directions. And, no surprise, being complex differentiable is much stronger because, once f has a complex derivative, then (being analytic) it has derivatives of all orders.

Notwithstanding the difference between the real and complex derivatives, the basic "rules" such as the Chain Rule or Inverse Function Rule remain in force. This yields proofs of Lemmas 45.8 and 45.9 by basic calculus (and the property (45.38).) We refer the reader to any basic textbook or a course on complex analysis where these topics are treated in detail.