43. THE METRIC SPACE OF CONTINUOUS FUNCTIONS

We have previously dealt with uniform convergence from the perspective of individual functions. Here we will observe that these notions can be categorized using metric space convergence. This point of view brings a number of useful consequences to the whole picture, including a functional-analytic version of the Arzelà-Ascoli Theorem.

We start with a definition:

Definition 43.1 *Given metric spaces X and Y, we write*

$$C(X,Y) := \{ f \in Y^X : \mathrm{Dom}(f) = X \wedge \text{ continuous}\} \tag{43.1}$$

and

$$C_{\mathrm{b}}(X,Y) := \{ f \in Y^X : \mathrm{Dom}(f) = X \wedge \text{ continuous} \wedge \text{ bounded}\} \tag{43.2}$$

for the space of continuous, resp., bounded and continuous functions $X \to Y$.

The key point for considering the space of bounded functions is:

Lemma 43.2 *Let (X,ρ_X) and (Y,ρ_Y) be metric spcaes. For $f,g \in C_{\mathrm{b}}(X,Y)$, let*

$$\rho_\infty(f,g) := \sup_{x\in X} \rho_Y\big(f(x),g(x)\big) \tag{43.3}$$

Then ρ_∞ is a metric on $C_{\mathrm{b}}(X,Y)$. If $Y = \mathbb{R}$ and ρ_Y is such that $\forall y,\tilde{y} \in \mathbb{R}\colon\ \rho_Y(y,\tilde{y}) = |y-\tilde{y}|$ then $\forall f,g \in C_{\mathrm{b}}(X,\mathbb{R})\colon\ \rho_\infty(f,g) = \|f-g\|$ where

$$\|f\| := \sup_{x\in X}\big|f(x)\big| \tag{43.4}$$

is a norm on $C_{\mathrm{b}}(X,\mathbb{R})$.

Proof. The symmetry, positivity and the triangle inequality are clear from the similar properties of ρ_Y. For the strict positivity we note that if $\rho_\infty(f,g) = 0$ then $\forall x \in X\colon f(x) = g(x)$ and so $f = g$ as desired. The corresponding properties when $(Y,\rho_Y) = (\mathbb{R},|\cdot|)$ are verified similarly. □

We will henceforth call ρ_∞ the *supremum metric* and $\|\cdot\|$ the *supremum norm*. These are generally defined only for bounded functions even though $\rho_\infty(f,g)$ may be finite even if both f and g are unbounded. The connection with the previously defined concepts is now as follows:

Lemma 43.3 *Let X and Y be metric spaces and $\{f_n\}_{n\in\mathbb{N}}$ and f functions from $C_{\mathrm{b}}(X,Y)$. Then*

$$f_n \to f \text{ uniformly } \Leftrightarrow\ f_n \to f \text{ in } \big(C_{\mathrm{b}}(X,y),\rho_\infty\big) \tag{43.5}$$

and

$$\{f_n\}_{n\in\mathbb{N}} \text{ uniformly Cauchy } \Leftrightarrow\ \{f_n\}_{n\in\mathbb{N}} \text{ Cauchy in } \big(C_{\mathrm{b}}(X,y),\rho_\infty\big) \tag{43.6}$$

Proof. This is checked directly from the definition of uniform convergence and the uniform-Cauchy property. □

As it turns out, additional properties imposed on the metric spaces X and Y will result in additional properties of the metric space $(C_{\mathrm{b}}(X,Y),\rho_\infty)$. For instance, we have:

Lemma 43.4 *Let (X,ρ_X) and (Y,ρ_Y) be metric spaces. Then*

$$(Y,\rho_Y)\text{ complete } \Rightarrow \big(C_b(X,y),\rho_\infty\big)\text{ complete} \tag{43.7}$$

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(C_b(X,Y),\rho_\infty)$. By Lemma 43.3, $\{f_n\}_{n\in\mathbb{N}}$ is uniformly Cauchy. Under the assumption that (Y,ρ_Y) is completely, Lemma 39.7 implies existence of $f\in C_b(X,Y)$ such that $f_n\to f$ uniformly. By Lemma 43.3, $f_n\to f$ in $(C_b(X,Y),\rho_\infty)$. □

Another observation concerns the space X:

Lemma 43.5 *Let (X,ρ_X) and (Y,ρ_Y) be metric spaces. Then*

$$(X,\rho_X)\text{ compact } \Rightarrow C(X,Y)=C_b(X,y) \tag{43.8}$$

Proof. By a corollary to Bolzano-Weierstrass Theorem, a continuous functions on a compact metric space is bounded. □

The metric space point of view is useful as it allows us to apply general facts from metric space theory to the specific set of continuous functions. We have already encountered an example of this in the proof of Lemma 42.5. Another example is:

Lemma 43.6 *Let (X,ρ_X) be a metric space and let $a\in X$. For each $z\in X$, let*

$$f_z(x):=\rho_X(z,x)-\rho_X(a,x) \tag{43.9}$$

Then $f_z\in C_b(X,\mathbb{R})$ for all $z\in X$ and

$$\forall z,\tilde z\in X\colon\ \|f_z-f_{\tilde z}\|=\rho_X(z,\tilde z) \tag{43.10}$$

In particular, $\phi(z):=f_z$ defines an isometry of (X,ρ_X) onto $\operatorname{Ran}(f)$*. The space $(\overline{\operatorname{Ran}(f)},\|\cdot\|)$ is a completion of (X,ρ_X).*

Proof. The triangle inequality shows

$$\big|f_z(x)\big|\leqslant\big|\rho_X(z,x)-\rho_X(a,x)\big|\leqslant\rho_X(z,a) \tag{43.11}$$

Equality occurs for $x:=a$ and so we get $\|f_z\|=\rho_X(a,z)$. In particular, f_z is Lipschitz continuous and bounded, i.e., $f_z\in C_b(X,\mathbb{R})$, for all $z\in X$. The proof of (43.10) follows exactly the same lines with $\tilde z$ instead of a.

Recall that a completion of (X,ρ_X) is a metric space $(\overline{X},\overline{\rho})$ for which there is an isometry $\phi\colon X\to\overline{X}$ such that $\operatorname{Ran}(\phi)$ is dense in $\overline{X}$. The density of $\operatorname{Ran}(f)$ in $\overline{\operatorname{Ran}}(f)$ is trivial so all we need to check that $(\overline{\operatorname{Ran}(f)},\|\cdot\|)$ is complete. This follows from the fact that $(C_b(X,\mathbb{R}),\|\cdot\|)$ is complete by Lemma 43.4 and the observation that a closed subset of a complete metric space is complete with respect to the relative metric. □

The simplicity of the previous proof is particularly striking when we recall how difficult was our original proof of this fact in 131AH. Indeed, there we built $\overline{X}$ as the set of equivalence classes of Cauchy sequences of elements from (X,ρ_X) while here we only appeal to elementary notions from metric spaces theory and (via Lemma 43.4) the completeness of $(\mathbb{R},|\cdot|)$. This is no surprise as our original argument was modeled on Cantor's proof of the completeness of the reals (and could even be used to construct the reals from the rationals).

We will now move to demonstrate the strength of these notions by giving a version of the Arzelà-Ascoli Theorem cast in the language of metric space of continuous functions with uniform metric:

Theorem 43.7 (Arzelà-Ascoli Theorem, functional form) *Let X and Y be metric spaces. Assume*

$$X \text{ compact} \wedge Y \text{ compact} \tag{43.12}$$

Then for all $F \subseteq C_{\mathrm{b}}(X,Y)$,

$$F \text{ is compact in } (C_{\mathrm{b}}(X,Y), \rho_\infty) \tag{43.13}$$

is equivalent to

$$F \text{ is closed, uniformly bounded and equicontinuous} \tag{43.14}$$

Here the word "closed" still refers to the metric space $(C_{\mathrm{b}}(X,Y), \rho_\infty)$.

Proof. We start by proving that (43.13) implies (43.14). Assume that F is compact in $(C_{\mathrm{b}}(X,Y), \rho_\infty)$. Then F is closed and bounded in $(C_{\mathrm{b}}(X,Y), \rho_\infty)$, where the latter means that there exists $f \in F$ such that $\sup_{h \in F} \rho_\infty(f,h) < \infty$. The fact that f is bounded in turns means that there exists $y \in Y$ such that $\sup_{x \in X} \rho_Y(f(x),y) < \infty$. But then

$$r := \sup_{h \in F} \rho_Y\big(y, h(x)\big) \leqslant \sup_{h \in F} \rho_\infty(f,h) + \sup_{x \in X} \rho_Y(f(x),y) < \infty \tag{43.15}$$

and we then have $\operatorname{Ran}(h) \subseteq B_Y(y,r)$ for all $h \in F$, implying that F is uniformly bounded in accord with Definition 41.1.

In order to prove that F is equicontinuous, note that compactness of F implies total boundedness. Given $\epsilon > 0$ there thus exist $n \in \mathbb{N}$ and $f_0, \dots, f_n \in F$ such that

$$F \subseteq \bigcup_{i=0}^{n} B(f_i, r) \tag{43.16}$$

where $B(f,r) := \{h \in C(X,Y) \colon \rho_\infty(h,f) < r\}$. Each f_i is continuous and, since X is compact, it is even uniformly continuous. Thus for each $i = 0, \dots, n$ there exists $\delta_i > 0$ such that'$\rho_X(x,\tilde{x}) < \delta$ implies $\rho_Y(f_i(x), f_i(\tilde{x})) < \epsilon$. Taking instead $\delta := \min_{i=0,\dots,n} \delta_i$, we may assume that all δ_i's are equal to δ. Given $f \in \mathcal{F}$, and setting $i_0 := \min\{i = 0, \dots, n \colon f \in B(f_i, r)\}$, for any $x_0, x \in X$ with $\rho_X(x,x_0) < \delta$ we then have

$$\begin{aligned} \rho_Y\big(f(x), f(x_0)\big) &\leqslant \rho_Y\big(f(x), f_{i_0}(x)\big) \\ &\quad + \rho_Y\big(f_{i_0}(x), f_{i_0}(x_0)\big) + \rho_Y\big(f(x_0), f_{i_0}(x_0)\big) < 3\epsilon \end{aligned} \tag{43.17}$$

This applies to all $f \in F$ and x_0 so F is equicontinuous as desired.

Let us now move to the opposite implication; namely, that (43.14) implies (43.13). Let $F \subseteq C(X,Y)$ be closed, uniformly bounded and equicontinuous. Let $\{f_n\}_{n \in \mathbb{N}} \in F^{\mathbb{N}}$. Since the functions in this sequence are equicontinuous the Arzelà-Ascoli Theorem (Theorem 42.3) shows the existence of a subsequence $\{f_{n_k}\}_{k \in \mathbb{N}}$ and a function f such that $f_{n_k} \to f$ uniformly. Since F is closed, $f \in F$. It follows that F is sequentially compact and hence (assuming the Axiom of Choice), compact. □

We remark that, in class, the above was stated under the weaker assumption that Y is locally compact. This turns out to be incorrect because the most standard definition of this notion is different than what was stated in class:

Definition 43.8 (Local compactness) *A metric space* (X, ρ_X) *is said to be* locally compact *if for each* $x \in X$ *there is* $r > 0$ *such that the closure of* $B(x, r)$ *is compact.*

What the argument given in class gave was that it suffices that Y has the Heine-Borel property; namely, that all bounded closed subsets of Y are compact. This is stronger than local compactness but not by my in practical sense. Indeed, one can always make the Heine-Borel property fail in a a locally compact space if we re-metrize the space by a bounded metric. What matters more is that the above works for Y being the reals endowed with the Euclidean metric.

We also note that, while our proof of Theorem 43.7 relied on the equivalence between sequential compactness and compactness, this can be avoided.