41. EQUICONTINUITY

The purpose of this section is to explore further the relation between uniform convergence and continuity. In particular, we will prove a number of results developed independently by C. Arzelà and G. Ascoli in the late 19th century. This theory culminates in a well-known theorem that now bears Arzelà and Ascoli's names which we will state and prove in the next section.

41.1 Necessary conditions for uniform convergence.

A goal of Arzelà and Ascoli's theory is to identify necessary and sufficient conditions for uniform convergence. Such an endeavor usually begins with finding a good number of necessary conditions and then noting that a suitable subset of these can be shown to be also sufficient. We start with a definition:

Definition 41.1 *A function $f\colon A \to Y$ from a set A to a metric space (Y, ρ_Y) is said to be* bounded *if there exists $y \in Y$ and $r > 0$ such that*

$$\exists y \in Y \, \exists r > 0\colon \ \operatorname{Ran}(f) \subseteq B_Y(y, r) = \big\{y' \in X\colon \rho_X(y, y') < r\big\} \tag{41.1}$$

A family $\{f_\alpha\colon \alpha \in I\}$ of functions is said to be uniformly bounded *if*

$$\exists y \in Y \, \exists r > 0 \, \forall \alpha \in I\colon \ \operatorname{Ran}(f_\alpha) \subseteq B_Y(y, r). \tag{41.2}$$

We then have:

Lemma 41.2 *Let $\{f_n\}_{n\in\mathbb{N}}$ and f be functions $X \to Y$ for metric spaces X and Y. Then*

$$\begin{aligned} &f_n \to f \text{ uniformly} \wedge f \text{ bounded} \\ &\qquad\qquad \Rightarrow \ \exists n_0 \in \mathbb{N}\colon \ \{f_n\colon n \geqslant n_0\} \text{ uniformly bounded} \end{aligned} \tag{41.3}$$

Proof. Let ρ_Y be the metric of Y and let $y \in Y$. Then

$$\sup_{x\in X} \rho_Y\big(y, f_n(x)\big) \leqslant \sup_{x\in X} \rho_Y\big(y, f(x)\big) + \sup_{x\in X} \rho_Y\big(f_n(x), f(x)\big) \tag{41.4}$$

If f is bounded, then the first supremum on the right is finite. The uniform convergence $f_n \to f$ in turn implies that the second suprmum is eventually less than, say, one. $\square$

Note, however, that uniform conditions can take place without the functions being bounded. Indeed, $f_n(x) := x + 1/n$ converges uniformly to $f(x) := x$ yet none of these functions (as functions on $\mathbb{R}$) are bounded.

A more interesting necessary conditions arises when we assume the limit function to be continuous:

Lemma 41.3 *Let X and Y be metric spaces and $\{f_n\}_{n\in\mathbb{N}}$ and f functions $X \to Y$. Suppose $f_n \to f$ uniformly and assume $x_0 \in X$ is such that f is continuous at x_0. Then*

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \exists n_0 \in \mathbb{N} \, \forall n \geqslant n_0\colon \ \sup_{x\in B_X(x,\delta)} \rho_Y\big(f_n(x), f_n(x_0)\big) < \epsilon \tag{41.5}$$

If, in addition, f_n is continuous for each $n \in \mathbb{N}$, then the above holds with $n_0 := 0$.

Proof. Suppose f is continuous at x_0. Then, given $\epsilon > 0$, there exists $\delta > 0$ such that $\rho_X(x, x_0) < \delta$ implies $\rho_Y(f(x), f(x_0)) < \epsilon$. The uniform convergence in turn gives $n_0 \in \mathbb{N}$

such that $\rho_Y(f_n(x), f(x)) < \epsilon$ for all $n \geqslant n_0$ and all $x \in X$. Putting this together, for $x \in X$ with $\rho_X(x, x_0) < \delta$ and $n \geqslant n_0$ we thus get

$$\begin{aligned}\rho_Y\bigl(f_n(x), f_n(x_0)\bigr) &\leqslant \rho_Y\bigl(f_n(x), f(x)\bigr) \\ &\qquad + \rho_Y\bigl(f(x), f(x_0)\bigr) + \rho_Y\bigl(f(x_0), f_n(x_0)\bigr) < 3\epsilon\end{aligned} \tag{41.6}$$

Relabeling ϵ this gives (41.5). For the second part of the claim, the continuity of f_n allow us to choose $\delta_n > 0$ such that $\rho_X(x, x_0) < \delta$ implies $\rho_Y(f_n(x), f_n(x_0)) < \epsilon$. With n_0 as above, we then take $\delta := \min_{0 \leqslant k \leqslant n_0} \delta_n$ and check that the inequality (41.5) is valied (with 3ϵ instead of ϵ) for all $n \in \mathbb{N}$. □

The property (41.5) states that, except for a finite number of n's, all f_n vary very little on a small neighborhood of x_0. We will give this property a name:

Definition 41.4 (Equicontinuity) *Given metric spaces (X, ρ_X) and (Y, ρ_Y) and $x_0 \in X$, we say that a family $\{f_\alpha \colon \alpha \in I\}$ of functions $X \to Y$ is* equicontinuous *at x_0 if*

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall x \in X \colon\; \rho_X(x, x_0) < \delta \;\Rightarrow\; \sup_{\alpha \in I} \rho_Y\bigl(f_\alpha(x), f_\alpha(x_0)\bigr) < \epsilon. \tag{41.7}$$

We say that $\{f_\alpha \colon \alpha \in I\}$ is equicontinuous *if it is equicontinuous at all $x_0 \in X$.*

We can even strengthen the observation in Lemma 41.3 to the form that does not refer to the limit function:

Lemma 41.5 *Let X and Y be metric spaces and $\{f_n\}_{n \in \mathbb{N}}$ a sequence of functions $X \to Y$. Let $x_0 \in X$. Then*

$$\begin{aligned}\{f_n\}_{n \in \mathbb{N}} \text{ uniformly Cauchy } \wedge \bigl(\forall n \in \mathbb{N} \colon f_n \text{ continuous at } x_0\bigr) \\ \Rightarrow \{f_n\}_{n \in \mathbb{N}} \text{ equicontinuous at } x_0\end{aligned} \tag{41.8}$$

Proof. Given $\epsilon > 0$, the uniform Cauchy property shows the existence of $n_0 \in \mathbb{N}$ such that $n \geqslant n_0$ implies $\rho_X(f_n(x), f_{n_0}(x)) < \epsilon$. Since f_{n_0} is continuous, we can now run the argument in the proof of Lemma 41.3 with f replaced by f_{n_0}. □

A key reason for introducing equicontinuity is that it suffices to ensure that pointwise convergent sequence of functions has a continuous limit:

Lemma 41.6 *Let X and Y be metric spaces and $\{f_n\}_{n \in \mathbb{N}}$ a sequence of functions $X \to Y$. Suppose that*

$$\forall x \in X \colon\; f(x) := \lim_{n \to \infty} f_n(x) \text{ exists} \tag{41.9}$$

Then

$$\{f_n\}_{n \in \mathbb{N}} \text{ equicontinuous } \Rightarrow\; f \text{ continuous} \tag{41.10}$$

Proof. Let $x_0 \in X$ and, given $\epsilon > 0$, let $\delta > 0$ be such that $\rho_X(x, x_0) < \delta$ implies that $\forall n \in \mathbb{N} \colon\; \rho_Y(f_n(x), f_n(x_0)) < \epsilon$. Passing to $n \to \infty$, we then get that $\rho_X(x, x_0) < \delta$ then implies $\rho_Y(f(x), f(x_0)) \leqslant \epsilon < 2\epsilon$. Hence f is continuous at x_0. □

An interesting twist to the previous lemma is that, under equicontinuity, we do not even need that the pointwise convergence takes place everywhere:

Lemma 41.7 *Let X and Y be metric spaces and $\{f_n\}_{n \in \mathbb{N}}$ a sequence of functions $X \to Y$ with $\forall n \in \mathbb{N} \colon\; Dom(f_n) = X$. Suppose that*

(1) $\{f_n\}_{n\in\mathbb{N}}$ *is equicontinuous,*
(2) *there exists* $A \subseteq X$ *such that*

$$A \text{ dense in } X \wedge \forall x \in A\colon \lim_{n\to\infty} f_n(x) \text{ exists} \tag{41.11}$$

(3) Y *is complete.*

Then there exists a continuous $f\colon X \to Y$ *with* $Dom(f) = X$ *such that* $f_n \to f$ *pointwise.*

Proof. Let A be as in the statement and let $x_0 \in X$. Pick $\epsilon > 0$. The equicontinuity then gives us $\delta > 0$ such that $\rho_X(x,x_0) < \delta$ implies $\forall n \in \mathbb{N}\colon \rho_Y(f_n(x), f_n(x_0)) < \epsilon$. The density of A in turn gives an $x \in A$ with $\rho_X(x,x_0) < \delta$ and the convergence (41.11) gives us $n_1 \in \mathbb{N}$ such that $m,n \geqslant n_1$ implies $\rho_Y(f_n(x), f_m(x)) < \epsilon$. The 3ϵ-argument now shows that, for all $m,n \geqslant \max\{n_0,n_1\}$

$$\begin{aligned}\rho_Y\bigl(f_m(x_0), f_n(x_0)\bigr) &\leqslant \rho_Y\bigl(f_m(x), f_m(x_0)\bigr)\\ &\qquad + \rho_Y\bigl(f_m(x), f_n(x)\bigr) + \rho_Y\bigl(f_n(x), f_n(x_0)\bigr) < 3\epsilon\end{aligned} \tag{41.12}$$

We have thus shown that $\{f_n(x_0)\}_{n\in\mathbb{N}}$ is Cauchy in (Y,ρ_Y). Since Y is assumed to be complete, $f(x_0) := \lim_{n\to\infty} f_n(x_0)$ exists for all $x_0 \in X$. The limit function is continuous by Lemma 41.6. □

41.2 Theorems of Arzelà and Ascoli.

Having demonstrated usefulness of equicontinuity, let us give some examples of equicontinuous families of functions. A typical example are k-th order polynomials

$$\Bigl\{x \mapsto \sum_{i=0}^{k} a_k x^k\colon a_0 \in \mathbb{R} \wedge a_1,\dots,a_k \in [-1,1]\Bigr\} \tag{41.13}$$

Note that, while the value of a_0 need not be constrained because it does not enter differences of function, the other coefficients must be constrained for otherwise we cannot find one δ to fit the same ϵ. Moving back to the general setting, the family of Lipschitz continuous functions

$$\Bigl\{f \in Y^X\colon \bigl(\forall x,\tilde{x} \in X\colon \rho_Y(f(x), f(\tilde{x})) \leqslant \lambda \rho_X(x,\tilde{x})\bigr)\Bigr\} \tag{41.14}$$

with the Lipschitz constant bounded by $\lambda > 0$ is equicontinuous. (Indeed, here δ is chosen from ϵ by $\delta := \epsilon/\lambda$.) The same argument works if we replace Lipschitz continuity by Hölder continuity or, even more generally, by forcing a particular modulus of continuity as in

$$\Bigl\{f \in Y^X\colon \bigl(\forall x,\tilde{x} \in X\colon \rho_Y(f(x), f(\tilde{x})) \leqslant h\bigl(\rho_X(x,\tilde{x})\bigr)\bigr)\Bigr\} \tag{41.15}$$

where $h\colon [0,\infty) \to [0,\infty)$ is a non-decreasing function such that $\lim_{t\to 0^+} h(t) = 0$.

These examples may seem special, but imposing a particular modulus of continuity is actually a typical way equicontinuity is verified in practice. As applications of the above observations, from the Mean-Value Theorem we get that

$$\Bigl\{f \in \mathbb{R}^{[a,b]}\colon \text{continuous on } [a,b] \wedge \text{differentiable on } (a,b) \wedge |f'| \leqslant M\Bigr\} \tag{41.16}$$

and, by (31.27), also

$$\left\{x \mapsto \int_a^x f(t)\mathrm{d}t\colon\; f \in \mathbb{R}^{[a,b]} \;\wedge\; \mathrm{RI} \;\wedge\; \sup_{t\in[a,b]}\bigl|f(t)\bigr| < M\right\} \tag{41.17}$$

are equicontinuous families of functions $[a,b] \to \mathbb{R}$ for each $M > 0$. (We leave easy proofs of these facts to the reader.)

These above were all positive examples that reflect on various "typical" situations in which equicontinuity is used and is useful. However, once we enter the setting of general metric spaces, more singular examples of equal interest can be produced. For instance, endowing the rationals with the Euclidean metric, consider functions $f_n\colon \mathbb{Q} \cap (0,\infty) \to \mathbb{R}$ defined by

$$f_n(x) := \begin{cases} 0, & \text{if } nx < \sqrt{2},\\ 1, & \text{otherwise.}\end{cases} \tag{41.18}$$

Then f_n is continuous for each $n \in \mathbb{N}$ and, in fact, the family $\{f_n\}_{n\in\mathbb{N}}$ is equicontinuous with $f_n \to 1$ (on $\mathbb{Q} \cap (0,1)$) as $n \to \infty$. Yet, importantly, the convergence is not uniform because $\mathrm{Ran}(f_n) = \{0,1\}$ for all $n \in \mathbb{N}$.

The previous example demonstrates the need for an additional assumption in order to "upgrade" a pointwise-convergent equicontinuous sequence to one that converges uniformly. This assumption, which constitutes perhaps the most important observation of the work of Arzelà and Ascoli, turns out to be that of compactness. Here is a first result in this vain:

Theorem 41.8 *Let X and Y be metric spaces and $\{f_n\}_{n\in\mathbb{N}}$ and f functions $X \to Y$. Assume*

$$\{f_n\}_{n\in\mathbb{N}} \text{ equicontinuous } \;\wedge\; f_n \to f \text{ pointwise } \;\wedge\; X \text{ compact} \tag{41.19}$$

Then $f_n \to f$ uniformly.

Proof. The idea of the proof is simple: We use equicontinuity and compactness to effectively represent the sequence $\{f_n\}_{n\in\mathbb{N}}$ by its values at a finite set of points. Then we note that, for functions on a finite set, pointwise and uniform convergence are equivalent.

Let $\epsilon > 0$ and, for each $x \in X$, let

$$\delta(x) := \frac{1}{2}\sup\Bigl\{\delta \in (0,1)\colon \Bigl(\forall n \in \mathbb{N}\colon\; f_n\bigl(B_X(x,\delta)\bigr) \subseteq B_Y\bigl(f(x),\epsilon\bigr)\Bigr)\Bigr\}. \tag{41.20}$$

The assumed equicontinuity ensures that the set on the right is non-empty; the extra factor of $1/2$ in the front in turn guarantees that

$$\forall x \in X\, \forall n \in \mathbb{N}\colon\;\; \delta(x) > 0 \;\wedge\; f_n\bigl(B_X(x,\delta(x))\bigr) \subseteq B_Y\bigl(f_n(x),\epsilon\bigr) \tag{41.21}$$

(Many texts will simply "choose" $\delta(x)$ this way, but this requires Axiom of Choice that our definition avoids.) Since the sets $\{B_X(x,\delta(x))\colon x \in X\}$ form an open cover of X, the compactness ensures the existence of $K \in \mathbb{N}$ and $x_0,\dots,x_k \in X$ such that

$$X = \bigcup_{i=0}^{K} B_X\bigl(x_i,\delta(x_i)\bigr) \tag{41.22}$$

The fact that $f_n(x_i) \to f(x_i)$ in turn implies that $\{f_n(x_i)\}_{n\in\mathbb{N}}$ is Cauchy meaning that

$$\forall i = 0,\dots,K\, \exists N_i \in \mathbb{N}\, \forall m,n \geqslant N_i\colon\;\; \rho_Y\bigl(f_n(x_i),f_m(x_i)\bigr) < \epsilon \tag{41.23}$$

Let $N := \max_{i=0,\dots,K} N_i$. Then for all $z \in X$ and i such that $z \in B_X(x_i, \delta(x_i))$, (41.21) implies

$$\forall n \in \mathbb{N}\colon\ \rho_Y\big(f_n(z), f_n(x_i)\big) < \epsilon \tag{41.24}$$

and so, for all $n, m \geqslant N$, we thus get

$$\begin{aligned}\rho_Y\big(f_n(z), f_m(z)\big) &\leqslant \rho_Y\big(f_n(z), f_n(x_i)\big)\\ &\qquad + \rho_Y\big(f_n(x_i), f_m(x_i)\big) + \rho_Y\big(f_m(x_i), f_m(z)\big) < 3\epsilon.\end{aligned} \tag{41.25}$$

It follows that

$$\forall n, m \geqslant N\colon\ \sup_{z \in X} \rho_Y\big(f_n(z), f_m(z)\big) \leqslant 3\epsilon \tag{41.26}$$

which, in light of our arbitrary choice of ϵ means that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly Cauchy. Since $f_n \to f$ pointwise, Lemma 39.7 shows that $f_n \to f$ uniformly. □

A natural question to ask is what can we say if the domain of the functions is not compact. The short answer is that, not much, as seen in the following examples: Given any metric space X, the sequence $1_{B_X(x_0,n)}$ converges to 1 pointwise but that convergence is uniformly only if X is bounded. We thus need to assume that X is bounded. But even that is not enough. Indeed, if for some $r > 0$, a metric space X is not covered by a finite number of balls of radius r, then we can pick a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n+1} \notin A_n := \bigcup_{i=0}^n B_X(x_i, r)$ for each $n \in \mathbb{N}$. The sequence 1_{A_n} then obeys $1_{A_n} \to 1$ pointwise but, since 1_{A_n} is always one somewhere, not uniformly.

It thus appears that the minimal assumption we have to make is that of total boundedness. However, as the example (41.18) shows, even that does not quite work unless we assume more than just plain equicontinuity. This is the content of:

Definition 41.9 (Uniform equicontinuity) *Given metric spaces (X, ρ_X) and (Y, ρ_Y), we say that a family $\{f_\alpha\colon \alpha \in I\}$ of functions $X \to Y$ is* uniformly equicontinuous *if*

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall x, y \in X\colon\ \rho_X(x, y) < \delta \Rightarrow \sup_{\alpha\in I} \rho_Y\big(f_\alpha(x), f_\alpha(y)\big) < \epsilon. \tag{41.27}$$

Comparing this with Definition 41.1, the key difference is that, given $\epsilon > 0$, the same $\delta > 0$ is now supposed to work for all points of x. Therefore, the same way that

$$\{f_\alpha\colon \alpha \in I\} \text{ equicontinuous } \Rightarrow\ \forall \alpha \in I\colon f_\alpha \text{ continuous} \tag{41.28}$$

we get

$$\{f_\alpha\colon \alpha \in I\} \text{ uniformly equicontinuous } \Rightarrow\ \forall \alpha \in I\colon f_\alpha \text{ uniformly continuous} \tag{41.29}$$

A corollary of (and, in fact, the original motivation for) the Bolzano-Weierstrass Theorem is the implication that f continuous on a compact set $\Rightarrow$ f uniformly continuous. It will thus not surprise us that the same applies even when continuity is replaced by equicontinuity:

Lemma 41.10 *Given metric spaces X and Y, let $\{f_\alpha\colon \alpha \in I\}$ be functions $X \to Y$. Then*

$$\begin{aligned}&\{f_\alpha\colon \alpha \in I\} \text{ equicontinuous} \wedge X \text{ compact}\\ &\qquad\qquad \Rightarrow\ \{f_\alpha\colon \alpha \in I\} \text{ uniformly equicontinuous}\end{aligned} \tag{41.30}$$

Proof (idea). Repeat the proof of Lemma 25.6 replacing f by f_n and noting that all properties hold uniformly in n. $\square$

With uniform equicontinuity in place of equicontinuity, we can "upgrade" Theorem 41.8 to the following form:

Theorem 41.11 *Let X and Y be metric spaces and $\{f_n\}_{n\in\mathbb{N}}$ and f functions $X \to Y$. Assume*

$$\{f_n\}_{n\in\mathbb{N}} \text{ uniformly equicontinuous } \wedge f_n \to f \text{ pointwise } \wedge X \text{ totally bounded} \tag{41.31}$$

Then $f_n \to f$ uniformly.

Proof. One way to prove this is to invoke the notion of a completion $\overline{X}$ of X while noting that uniformly equicontinuous families on X extend uniquely to uniformly equicontinuous families on $\overline{X}$. Since a completion of a totally bounded space is compact, the statement then follows from Lemma 41.7 and Theorem 41.8. While that perhaps explains better the choice of the assumptions, it will be easier to proceed by a direct argument.

Fix $\epsilon > 0$ and let $\delta > 0$ be such that $\rho_X(x, \tilde{x}) < \delta$ implies $\rho_Y(f_n(x), f_n(\tilde{x})) < \epsilon$. Now use total boundedness to find $K \in \mathbb{N}$ and $x_0, \dots, x_K \in X$ such that $X = B_X(x_i, \delta)$. Finally, let $N \in \mathbb{N}$ be such that $m, n \geqslant N$ implies $\rho_Y(f_n(x_i), f_m(x_i))$ for all $i = 0, \dots, K$. Then for $z \in X$ and $i = 0, \dots, K$ such that $z \in B_X(x_i, \delta)$, the condition $m, n \geqslant N$ implies

$$\begin{aligned} \rho_Y\bigl(f_n(z), f_m(z)\bigr) \leqslant \rho_Y\bigl(f_n(z), f_n(x_i)\bigr) \\ + \rho_Y\bigl(f_n(x_i), f_m(x_i)\bigr) + \rho_Y\bigl(f_m(x_i), f_m(z)\bigr) < 3\epsilon \end{aligned} \tag{41.32}$$

showing that $\{f_n\}_{n\in\mathbb{N}}$ is uniformly Cauchy. Since $f_n \to f$ pointwise, Lemma 39.7 gives that $f_n \to f$ uniformly. $\square$

We leave it to the reader to check that a uniform limit of uniformly equicontinuous functions is uniformly continuous.