40. Applications of uniform convergence

We will now move to some more advanced applications of uniform convergence to integration and differentiation theory. We will then apply these to power series and use them to finally define a number of important transcendental functions.

40.1 Integration and differentiation.

Our first application is to convergence of Riemann integrals. Recall that the Osgood Bounded Convergence Theorem (cf Theorem 38.2) states that if $\{f_n\}_{n\in\mathbb{N}}$ are uniformly bounded functions $[a,b]\to\mathbb{R}$ and $f\colon[a,b]\to\mathbb{R}$ is such that $f_n\to f$ pointwise *and* f is Riemann integrable, then $\int_a^b f_n(x)\mathrm{d}x\to\int_a^b f(x)\mathrm{d}x$. As it turns out, the statement is easier to make once we replace pointwise convergence by uniform convergence:

Theorem 40.1 *Let $a<b$ be reals and $\{f_n\}_{n\in\mathbb{N}}$ and f functions $[a,b]\to\mathbb{R}$ such that*

$$\Big(\forall n\in\mathbb{N}\colon f_n\text{ Riemann integrable}\Big)\ \wedge\ f_n\to f\text{ uniformly} \tag{40.1}$$

Then

$$f\text{ Riemann integrable}\ \wedge\ \lim_{n\to\infty}\int_a^b f_n(x)\mathrm{d}x=\int_a^b f(x)\mathrm{d}x \tag{40.2}$$

Proof. Notice that the additivity of the integral and Lemma 31.9 give

$$\left|\int_a^b f_m(x)\mathrm{d}x-\int_a^b f_n(x)\mathrm{d}x\right|=\left|\int_a^b (f_n-f_m)(x)\mathrm{d}x\right|\leqslant(b-a)\sup_{x\in[a,b]}\big|f_n(x)-f_m(x)\big| \tag{40.3}$$

By Lemma 39.7, $f_n\to f$ implies that the right-hand side is smaller than $\epsilon>0$ once m and n are sufficiently large. It follows that

$$\left\{\int_a^b f_n(x)\mathrm{d}x\right\}_{n\in\mathbb{N}}\text{ is Cauchy} \tag{40.4}$$

and since this is a real-valued sequence and $\mathbb{R}$ is complete,

$$L:=\lim_{n\to\infty}\int_a^b f_n(x)\mathrm{d}x\ \text{ exists.} \tag{40.5}$$

In particular, given $\epsilon>0$, there is $n_0\in\mathbb{N}$ such that

$$\forall n\geqslant n_0\colon\ \left|\int_a^b f_n(x)\mathrm{d}x-L\right|<\epsilon. \tag{40.6}$$

The uniform convergence in turn shows existence of $n_1\in\mathbb{N}$ such that

$$\forall n\geqslant n_1\colon\ \sup_{x\in[a,b]}\big|f_n(x)-f(x)\big|<\frac{\epsilon}{b-a}. \tag{40.7}$$

Set $m:=\max\{n_0,n_1\}$. The assumed Riemann integrability of f_m implies that there exists $\delta>0$ such that for any marked partition Π of $[a,b]$,

$$\|\Pi\|<\delta\ \Rightarrow\ \left|R(f_m,\Pi)-\int_a^b f_m(x)\mathrm{d}x\right|<\epsilon \tag{40.8}$$

The linearity of $f \mapsto R(f,\Pi)$ in turn gives

$$\bigl|R(f,\Pi)-R(f_m,\Pi)\bigr| \leqslant (b-a)\sup_{x\in[a,b]}\bigl|f_m(x)-f(x)\bigr| < \epsilon \tag{40.9}$$

where the second inequality follows from (40.7) and the fact that $m \geqslant n_1$. The triangle inequality then shows that, for each marked partition Π of $[a,b]$ with $\|\Pi\| < \delta$, we have

$$\begin{aligned}\bigl|R(f,\Pi)-L\bigr| &\leqslant \bigl|R(f,\Pi)-R(f_m,\Pi)\bigr| \\ &\quad+\left|R(f_m,\Pi)-\int_a^b f_m(x)\,\mathrm{d}x\right| + \left|\int_a^b f_m(x)\,\mathrm{d}x - L\right| < 3\epsilon\end{aligned} \tag{40.10}$$

thus proving that f is Riemann integrable and the integral of f equals L. □

The next application is to differentiation:

Theorem 40.2 *Given real numbers $a < b$, let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of differentiable functions $(a,b)\to\mathbb{R}$ such that*

$$\exists x_0 \in (a,b)\colon\ \lim_{n\to\infty} f_n(x_0) \text{ exists} \tag{40.11}$$

and

$$\{f_n'\}_{n\in\mathbb{N}} \text{ is uniformly Cauchy} \tag{40.12}$$

Then there exists a differentiable function $f\colon (a,b)\to\mathbb{R}$ such that

$$f_n \to f \text{ uniformly } \wedge\ f_n' \to f' \text{ uniformly} \tag{40.13}$$

In particular, the derivative can be exchanged with the limit $n\to\infty$.

Proof. For each $n\in\mathbb{N}$ define $\phi_n\colon (a,b)\to\mathbb{R}$ by

$$\phi_n(x) := \begin{cases}\frac{f_n(x)-f_n(x_0)}{x-x_0}, & \text{if } x \neq x_0,\\ f_n'(x_0), & \text{if } x = x_0.\end{cases} \tag{40.14}$$

Since f_n is continuous (being differentiable) and the derivative at x_0 exists, ϕ_n is continuous on (a,b). Langrange's Mean-Value Theorem shows

$$\forall x\in(a,b)\,\exists\xi\in(a,b)\colon\ \phi_n(x) = f_n'(\xi) \tag{40.15}$$

(For $x = x_0$ we use directly the definition.) Since $\{f_n'\}_{n\in\mathbb{N}}$ is uniformly Cauchy, so is $\{\phi_n\}_{n\in\mathbb{N}}$. Lemma 39.7 shows that there exists a function $\phi\colon (a,b)\to\mathbb{R}$ such that $\phi_n\to\phi$ uniformly. Being a uniform limit of continuous functions, ϕ is continuous.

Define $f\colon (a,b)\to\mathbb{R}$ by

$$f(x) := \lim_{n\to\infty} f_n(x_0) + \phi(x)(x-x_0). \tag{40.16}$$

Since $f_n(x) = f_n(x_0)+\phi_n(x)(x-x_0)$, the uniform convergence $\phi_n\to\phi$ implies uniform convergence $f_n\to f$. Next pick $x\in(a,b)$ and observe that the above Mean-Value Argument implies that also the functions $\psi_n\colon (a,b)\to\mathbb{R}$ defined by

$$\psi_n(y) := \begin{cases}\frac{f_n(y)-f_n(x)}{y-x}, & \text{if } y\neq x,\\ f_n'(x), & \text{if } y = x,\end{cases} \tag{40.17}$$

converge uniformly to $\psi\colon (a,b)\to\mathbb{R}$ defined by

$$\psi(y) := \begin{cases} \frac{f(y)-f(x)}{y-x}, & \text{if } y\neq x,\\ f'(x), & \text{if } y=x. \end{cases} \tag{40.18}$$

By Corollary 39.5, this justifies the exchange of limits in the formula

$$\begin{aligned} \lim_{y\to x}\frac{f(y)-f(x)}{y-x} &= \lim_{y\to x}\lim_{n\to\infty}\frac{f_n(y)-f_n(x)}{y-x}\\ &= \lim_{n\to\infty}\lim_{y\to x}\frac{f_n(y)-f_n(x)}{y-x} = \lim_{n\to\infty} f_n'(x) \end{aligned} \tag{40.19}$$

proving that $f'(x)$ exists and $f_n'\to f$ pointwise. Since $\{f_n'\}_{n\in\mathbb{N}}$ is uniformly Cauchy, Corollary 39.8 implies that $f_n'\to f'$ uniformly. □

This theorem is usually applied in proofs that a function constructed as a limit is smooth. We will get to such examples momentarily, but before we do that let us demonstrate the power of the result by using it to construct a function that is *not* differentiable at any given finite or countable set of points:

Lemma 40.3 *Let $A\subseteq\mathbb{R}$ be finite or countable. Then there exists $f\colon\mathbb{R}\to\mathbb{R}$ bounded, differentiable with f' bounded yet such that*

$$\bigl\{x\in\mathbb{R}\colon f' \text{ NOT continuous at } x\bigr\} = A \tag{40.20}$$

Proof. Consider the function $h\colon\mathbb{R}\to\mathbb{R}$ defined by

$$h(x) := \begin{cases} \frac{x^2}{1+x^2}\sin(1/x), & \text{if } x\neq 0,\\ 0, & \text{if } x=0. \end{cases} \tag{40.21}$$

(While sin has not yet been defined, all we need that it is a a bounded differentiable function whose derivative, to be denoted cos, exists and is bounded on $\mathbb{R}$ but does not admit a limit at $\pm\infty$.) Then h is differentiable with

$$h'(x) := \begin{cases} -\frac{1}{1+x^2}\cos(1/x) - \frac{2x}{(1+x^2)^2}\sin(1/x), & \text{if } x\neq 0,\\ 0, & \text{if } x=0, \end{cases} \tag{40.22}$$

This h' is bounded and continuous on $\mathbb{R}\smallsetminus\{0\}$ but does not admit a limit as $x\to\infty$. Now pick A (to be assumed infinite countable for simplicity) and let $\{q_n\}_{n\in\mathbb{N}}$ enumerate its points. Then define $f\colon\mathbb{R}\to\mathbb{R}$ by

$$f(x) := \sum_{n=0}^{\infty} 2^{-n} h(x-q_n) \tag{40.23}$$

Since h and h' are bounded, Theorem 40.2 shows that f is differentiable with

$$f'(x) = \sum_{n=0}^{\infty} 2^{-n} h'(x-q_n) \tag{40.24}$$

where the series converges uniformly on $\mathbb{R}$.

For $x\neq A$, all functions under the sum are continuous are continuous at x and so Theorem 39.4 tells us that also f' is continuous at x. For $x=q_m$, the same holds except for the term corresponding to $n=m$, which is not continuous at q_m. Hence, nor is f'. □

40.2 Power series and transcendental functions.

We now move to the positive side of the story by demonstrating its application to functions given as a convergent power series. Theorem 40.1 allows us to integrate these term-by-term. However, more important is usually the corresponding statement for differentiation. The following is a direct consequence of Theorem 40.2:

Corollary 40.4 *Let $\{a_n\}_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ be such that*

$$R := \left[\limsup_{n\to\infty} |a_n|^{1/n}\right]^{-1} > 0 \tag{40.25}$$

with $+\infty^{-1} := 0$ and $0^{-1} := +\infty$. Let $x_0 \in \mathbb{R}$. Then $f\colon (x_0 - R, x_0 + R) \to \mathbb{R}$ defined by $f(x) := \sum_{n=0}^{\infty} a_n(x-x_0)^n$ is arbitrary many times differentiable with the k-th derivative satisfying

$$f^{(k)}(x) = \sum_{n=k}^{\infty}\left[\prod_{i=0}^{k-1}(n-i)\right] a_n (x-x_0)^n \tag{40.26}$$

where the series converges absolutely on $(x_0 - R, x_0 + R)$ and uniformly on any compact subinterval thereof.

Proof. Let $k \in \mathbb{N}$ obey $k \geqslant 1$. Following the proof of Lemma 39.10, for each $r \in (0, R)$ and $\epsilon \in (0, R - r)$ there exists $A \in (0, \infty)$ such that (39.39) holds. This means that

$$\forall x \in [x_0 - r, x_0 + r]\colon\ \left|\frac{\mathrm{d}^k}{\mathrm{d}x^k} a_n (x-x_0)^n\right| \leqslant A r^{-k} n^k \left(\frac{r}{R-\epsilon}\right)^n \tag{40.27}$$

The right-hand side is summable on n and so the series of k-th derivatives converges uniformly on $[x_0 - r, x_0 + r]$ by the Weierstrass M-test. The claim now follows by an inductive application of Theorem 40.2. □

We now use these to prove:

Lemma 40.5 (Exponential, sine and cosine) *The real-valued functions* exp, sin *and* cos *are well-defined for all $x \in \mathbb{R}$ by*

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{40.28}$$

$$\sin(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{40.29}$$

$$\cos(x) := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{40.30}$$

where the series converge absolutely everywhere and uniformly on any compact subinterval of $\mathbb{R}$. Moreover, these functions are arbitrary many times differentiable on $\mathbb{R}$ with the derivatives

$$\exp'(x) = \exp(x) \ \wedge\ \sin'(x) = \cos(x) \ \wedge\ \cos'(x) = -\sin(x) \tag{40.31}$$

at each $x \in \mathbb{R}$.

Proof. We claim that the quantity R from (40.25) equals $+\infty$ for all three series. To see this, note that for $n \geqslant 3$ we have $n/3 \geqslant 1$ and $n - \lceil n/3 \rceil \geqslant n - n/3 - 1 \geqslant n/3$ and so

$$n! \geqslant \prod_{n/3 \leqslant i \leqslant n} i \geqslant (n/3)^{n/3} \tag{40.32}$$

and so $|n!|^{1/n} \geqslant (n/3)^{1/3}$. It follows that $\limsup_{n\to\infty}(1/n!)^{1/n} = 0$ thus proving that the radius of convergence of all three series is infinite. Corollary 40.4 allows us to differentiate the series term-ty-term which then readily shows (40.31). □

The exp function, to be called *exponential*, obeys the following:

Lemma 40.6 *The function* exp *defined above obeys*

$$\forall x, y \in \mathbb{R}\colon\ \ \exp(x+y) = \exp(x)\cdot\exp(y) \tag{40.33}$$

In particular, setting $\mathrm{e} := \sum_{n=0}^{\infty} \frac{1}{n!}$ *we have*

$$\forall x \in \mathbb{R}\colon\ \ \exp(x) = \mathrm{e}^x \tag{40.34}$$

where $\mathrm{e}^x := \inf\{\sqrt[q]{\mathrm{e}^p}\colon\ p \in \mathbb{Z} \wedge q \geqslant 1 \wedge x \leqslant p/q\}$.

We leave the proof of the lemma to a homework exercise while noting that the main idea is the following fact: If $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and obeys $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ then either f vanishes identically everywhere, or there is $a > 0$ such that $f(x) = a^x$ for all $x \in \mathbb{R}$. The continuity assumption cannot be dropped; indeed, without that other solutions that are not of this form exist.

For the sine and cosine functions we in turn get:

Lemma 40.7 *The* sin *and* cos *functions defined above satisfy:*

(1) $\forall x \in \mathbb{R}\colon\ \sin(x)^2 + \cos(x)^2 = 1$ *and so* sin *and* cos *take values in* $[-1, 1]$,
(2) *the addition formulas hold:*

$$\forall x, y \in \mathbb{R}\colon\quad \begin{aligned} \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \end{aligned} \tag{40.35}$$

(3) *The number*

$$\pi := 2\inf\bigl\{t \geqslant 0\colon\ \cos(t) = 0\bigr\} \tag{40.36}$$

obeys $\pi \in (0, \infty)$ *and we have*

$$\forall x \in \mathbb{R}\colon\ \ \sin(x) = -\cos(x + \pi/2) = -\sin(x - \pi) \tag{40.37}$$

and so, in particular,

$$\forall x \in \mathbb{R}\colon\ \ \sin(x + 2\pi) = \sin(x) \ \wedge\ \cos(x + 2\pi) = \cos(x) \tag{40.38}$$

showing that sin *and* cos *are* 2π*-periodic.*

Note that the above defines two fundamental numbers in analysis; the Euler number e, named after early 18-th century Swiss mathematician and scientist Leonard Euler, and the Ludolphine number π, named after 16-th century German/Dutch mathematician Ludolph van Ceulen who computed π to 35 digits.

The definition $\mathrm{e} := \sum_{n=0}^{\infty} \frac{1}{n!}$ makes it easy to check that $\mathrm{e} \in (2, 3)$. For π we use that sin has to be increasing and non-negative on $[0, \pi/2]$ and, being equal to negative of its second derivative, also concave. This, along with its taking values in $[-1,1]$ shows $2x/\pi \leqslant \sin(x) \leqslant 1$. Integrating over $[0,\pi/2]$ then shows

$$\frac{\pi}{4} = \frac{x^2}{\pi}\Big|_0^{\pi/2} \leqslant \int_0^{\pi/2} \sin(x)\mathrm{d}x \leqslant x\Big|_0^{\pi/2} = \frac{\pi}{2} \tag{40.39}$$

which in light of $1 = \cos(0) - \cos(\pi/2) = \int_0^{\pi/2}\sin(x)\mathrm{d}x$ gives $\pi \in [2,4]$. If we instead use the upper bound $\sin(x) \leqslant x$, we even get $\pi^2 \geqslant 8$, i.e., $\pi \geqslant \sqrt{8}$.

The Euler and Ludoplhine numbers are known to extremely high accuracy:

$$\begin{aligned} \mathrm{e} &= 2.718281828459045235360287471352 7\dots \\ \pi &= 3.14159265358979323846264338327950288\dots \end{aligned} \tag{40.40}$$

Both numbers are irrational which, for e, is actually easy to show:

Lemma 40.8 $\mathrm{e} \notin \mathbb{Q}$

Proof. If e were rational, we could write it as $\mathrm{e} = p/q$ for some natural $p, q \geqslant 2$. Then $q!\mathrm{e}$ is an integer and, since $n!$ divides $q!$ when $n \leqslant q$, also

$$\sum_{n=q+1}^{\infty} \frac{q!}{n!} = q! - \sum_{n=0}^{q} \frac{q!}{n!} \tag{40.41}$$

is an integer. But $q \geqslant 2$ implies $q!/n! \geqslant \frac{1}{q+1}2^{-(n-q-1)}$ for $n \geqslant q+1$ and so

$$0 < \sum_{n=q+1}^{\infty} \frac{q!}{n!} \leqslant \sum_{n=q+1}^{\infty} \frac{1}{q+1}2^{-(n-q-1)} = \frac{1}{q+1}\sum_{k=0}^{\infty} 2^{-k} = \frac{2}{q+1} < 1 \tag{40.42}$$

a contradiction. So we must have $\mathrm{e} \notin \mathbb{Q}$ after all. □

A curious fact (discovered by L. Euler) is that these numbers are closely related. Indeed, the suspiciously similar form of the power series (40.28–40.30) is resolved into

$$\forall x \in \mathbb{R}\colon\ \mathrm{e}^{\mathrm{i}x} = \cos(x) + \mathrm{i}\sin(x) \tag{40.43}$$

where i is the imaginary number such that $\mathrm{i}^2 = -1$. (This requires defining these functions for complex-valued arguments; the proof that they converge is however identical.) In particular, we have

$$\mathrm{e}^{\mathrm{i}\pi} = -1 \tag{40.44}$$

The latter is merely a restatement of the definition of π (any odd multiple of π substituted for π still makes this TRUE), but the *Euler formula* (40.43) is quite important as it lays the basis of polar representation of complex numbers.

We will return to power series representation of functions (not just transcendental ones) when we discuss analytic functions. This will allow us to give an infinite series representation for π as well albeit not so rapidly convergent as the one of e that the argument from Lemma 40.8, which incidentally applies to any number of the form $\sum_{n=0}^{\infty}\frac{a_n}{n!}$ with $\{a_n\}_{n\in\mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$ of at most exponential growth, could be readily repeated.