38. Lebesgue and Henstock-Kurzweil integrals

Here we wrap up the discussion of Riemann's approach to integration. We first go over some shortcomings of Riemann integration theory and then offer glimpse into more advanced integration theories that address these better.

38.1 Shortcomings of the Riemann integral.

Let us return to the Riemann integral and review some of the advanced aspects, and shortcomings, of the theory. One unpleasant limitation of the Riemann integral is the requirement of boundedness, both for the integrated function and the underlying domain. This prevents us from integrating functions with divergences at the endpoints of integration domain directly, or to integrate over unbounded domains. This is usually fixed by taking more limits:

Definition 38.1 (Improper integral) *Let $f\colon (a,b] \to \mathbb{R}$ be Riemann integrable on $[a',b]$ for any $a' \in (a,b)$. The* improper Riemann integral *on $[a,b]$ is then defined by*

$$\lim_{a'\to a^+} \int_{a'}^{b} f(x)\mathrm{d}x \tag{38.1}$$

whenever the limit exists. A similar definition applies to the upper limit of integration and the limits $a \to -\infty$ and $b \to +\infty$.

Using this concept we can evaluate integrals

$$\int_0^1 \frac{1}{\sqrt{x}}\mathrm{d}x := \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\mathrm{d}x = \lim_{a\to 0^+} 2\sqrt{x}\Big|_a^1 = \lim_{a\to 0^+} 2[1-a] = 2 \tag{38.2}$$

where we use the FTCII along with the fact that $x \mapsto 2\sqrt{x}$ is an antiderivative of $x \mapsto \frac{1}{\sqrt{x}}$, as well as integrals such as

$$\begin{aligned} \int_0^1 \frac{\sin(1/x)}{x}\mathrm{d}x &:= \lim_{a\to 0^+} \int_a^1 x\frac{\mathrm{d}}{\mathrm{d}x}\cos(1/x)\mathrm{d}x \\ &= \lim_{a\to 0^+} x\cos(1/x)\Big|_a^1 - \lim_{a\to 0^+} \int_a^1 \cos(1/x)\mathrm{d}x \\ &= \cos(1) - \int_0^1 \cos(1/x)\mathrm{d}x \end{aligned} \tag{38.3}$$

where we used integration by parts and then the fact that $x \mapsto \cos(1/x)$ is Riemann integrable on $[0,1]$. Similarly, for the so called *Fresnel integral* we get

$$\begin{aligned} \int_0^\infty \sin(x^2)\mathrm{d}x &:= \lim_{b\to\infty} \int_0^b \sin(x^2)\mathrm{d}x = \lim_{b\to\infty} \int_0^b \frac{\sin(t)}{2\sqrt{t}}\mathrm{d}t \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} \int_{2\pi n}^{2\pi(n+1)} \frac{\sin(t)}{2\sqrt{t}}\mathrm{d}t, \end{aligned} \tag{38.4}$$

where use use the Substitution rule and then rewrote the general limit $b \to \infty$ as the limit along the naturals (this uses that the function decays to zero). Using the the fact that $\sin(t+\pi) = -\sin(t)$ we get

$$\int_{2\pi n}^{2\pi(n+1)} \frac{\sin(t)}{\sqrt{t}}\mathrm{d}t = \int_0^\pi \sin(t)\Bigl[\frac{1}{\sqrt{2\pi n+t}} - \frac{1}{\sqrt{2\pi n+\pi+t}}\Bigr]\mathrm{d}t. \tag{38.5}$$

Elementary manipulations now show

$$\Bigl|\frac{1}{\sqrt{2\pi n+t}} - \frac{1}{\sqrt{2\pi n+\pi+t}}\Bigr| \leqslant \frac{2\pi}{2(2\pi n)^{3/2}} = \frac{1}{\sqrt{8\pi}}\frac{1}{n^{3/2}}, \tag{38.6}$$

thus bounding the integral in (38.5) by $\frac{1}{\sqrt{8\pi}}n^{-3/2}$. It follows that the limit on the right of (38.4) exists. (The limit value of the integral is actually known to be $\sqrt{\pi/8}$.)

There is also a suitable limit procedure to address functions that have singularities inside the integration domain. Still, the need to take a limit (and exchange it with other limits if that is desired) means that improper integrals are sensitive to further manipulations and working with them is not always easy.

Another deficiency of the Riemann integral is the behavior of the integral when integrands converge to a function. As we noted after (37.25) and will expand on later, the Riemann integral is well adapted to uniform convergence, but not so much when the convergence is just pointwise. Indeed, as the example of the Dirichlet function shows, a pointwise limit of Riemann integrable functions need not be Riemann integrable. As it turns out, lack of integrability of the limit function is pretty much the only obstacle:

Theorem 38.2 (Osgood's Bounded Convergence Theorem) *Let $\{f_n\}_{n\in\mathbb{N}}$ be Riemann integrable functions on $[a,b]$ satisfying*

$$\exists c \in (0,\infty)\,\forall n\in\mathbb{N}\,\forall x\in[a,b]\colon\ \bigl|f_n(x)\bigr| \leqslant c \tag{38.7}$$

and

$$\forall x\in[a,b]\colon\ f(x) := \lim_{n\to\infty} f_n(x) \text{ exists} \tag{38.8}$$

Then

$$f \text{ Riemann integrable } \Rightarrow \lim_{n\to\infty}\int_a^b f_n(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x. \tag{38.9}$$

Proof. Assume that f is Riemann integrable and, for each $n\in\mathbb{N}$, define $h_n\colon[a,b]\to\mathbb{R}$ by $h_n(x) := f_n(x) - f(x)$. Then $\{h_n(x)\}_{n\in\mathbb{N}}$ are Riemann integrable with $\lim_{n\to\infty} h_n(x) = 0$. Our aim is to show that

$$\lim_{n\to\infty}\int_a^b h_n(x)\mathrm{d}x = 0 \tag{38.10}$$

We proceed by contradiction. If (38.10) fails, then $\int_a^b h_n(x)\mathrm{d}x$ and thus also $\int_a^b |h_n(x)|\mathrm{d}x$ stay outside an open interval containing the origin for infinitely many $n\in\mathbb{N}$. Assuming, without loss of generality, that this happens for all $n\in\mathbb{N}$, we need to show that

$$\eta := \inf_{n\in\mathbb{N}}\int_a^b \bigl|h_n(x)\bigr|\mathrm{d}x > 0 \tag{38.11}$$

is incompatible with $h_n(x)\to 0$ for all $x\in[a,b]$.

Assume (38.11) is TRUE. The definition of the integral implies that, for each $n \in \mathbb{N}$ there is $\delta_n > 0$ such that for each marked partition Π of $[a,b]$ with $\|\Pi\| < \delta_n$ we have

$$R\bigl(|h_n|,\Pi\bigr) \geqslant \frac{1}{2}\eta. \tag{38.12}$$

Replacing δ_n by $\min_{0\leqslant k\leqslant n}\delta_k$ we may assume that $\{\delta_n\}_{n\in\mathbb{N}}$ is non-increasing. Setting $m_n := \inf\{k \in \mathbb{N}\colon 2^k\delta_n > b-a\}$, denoting $t_i^n := a + 2^{-m_n}i(b-a)$ which we note obeys $t_i^n - t_{i-1}^n < \delta_n$ for all $i = 1,\dots,m_n$, and optimizing over the marked point in $[t_{i-1},t_i]$ turns (38.12) into

$$\forall n \in \mathbb{N}\colon \quad \sum_{i=1}^{m_n}\Bigl(\inf_{x\in[t_{i-1}^n,t_i^n]}\bigl|h_n(x)\bigr|\Bigr)(t_i^n - t_{i-1}^n) \geqslant \frac{\eta}{2} \tag{38.13}$$

Denote

$$I_n := \Bigl\{ i = 1,\dots,m_n\colon \inf_{x\in[t_{i-1}^n,t_i^n]}\bigl|h_n(x)\bigr| > \frac{\eta}{4(b-a)}\Bigr\} \tag{38.14}$$

Then

$$\begin{aligned}\sum_{i=1}^{m_n}&\Bigl(\inf_{x\in[t_{i-1}^n,t_i^n]}\bigl|h_n(x)\bigr|\Bigr)(t_i^n - t_{i-1}^n)\\ &\leqslant \Bigl(\sup_{x\in[a,b]}\bigl|h_n(x)\bigr|\Bigr)\sum_{i\in I_n}(t_i^n - t_{i-1}^n) + \frac{\eta}{4(b-a)}\sum_{i\notin I_n}(t_i^n - t_{i-1}^n)\end{aligned} \tag{38.15}$$

along with (38.13) and the fact that the last sum is at most $b-a$ show

$$\forall n \in \mathbb{N}\colon \quad \sum_{i\in I_n}(t_i^n - t_{i-1}^n) \geqslant \frac{\eta}{4+8c}, \tag{38.16}$$

where we used that $|h_n(x)| \leqslant 2c$ thanks to (38.7–38.8). We now need:

Lemma 38.3 (Arzelà) *Let $a < b$ be reals and, for each $n \in \mathbb{N}$, let $\{J_{n,k}\}_{k=0}^{r_n}$ be a finite collection of disjoint open intervals in (a,b) such that*

$$\exists \eta' > 0\,\forall n \in \mathbb{N}\colon \quad \sum_{k=0}^{r_n}\text{length}(J_{n,k}) \geqslant \eta' \tag{38.17}$$

Then

$$\bigcap_{n\in\mathbb{N}}\bigcup_{m\geqslant n}\bigcup_{k=0}^{r_n} J_{n,k} \neq \varnothing \tag{38.18}$$

Deferring the proof of this lemma until after this proof is finished, we apply it for the choices $r_n := |I_n|$ and $\{J_{n,k}\}_{k=0}^{r_n}$ being the intervals $\{(t_{i-1}^n,t_i^n)\colon i \in I_n\}$. The definition of I_n implies, for each $x \in [a,b]$,

$$x \in \bigcap_{n\in\mathbb{N}}\bigcup_{m\geqslant n}\bigcup_{i\in I_n}(t_{i-1}^n,t_i^n) \;\Rightarrow\; \limsup_{n\to\infty}\bigl|h_n(x)\bigr| \geqslant \frac{\eta}{4(b-a)} \tag{38.19}$$

and so, with the help of Lemma 38.3, (38.16) contradicts the assumption that $h_n(x) \to 0$ as $n \to \infty$ for all $x \in [a,b]$. It follows that (38.10) holds after all. □

It remains to give:

Proof of Lemma 38.3 (modulo facts from measure theory). We will give only the main steps of the proof assuming the following fact: If $\{J_i\}_{i\in\mathbb{N}}$ are (possibly empty) open intervals in (a,b) and $\{I_n\}_{n\in\mathbb{N}}$ are (possibly empty) disjoint open intervals in (a,b), then

$$\bigcup_{n\in\mathbb{N}} I_n \subseteq \bigcup_{i\in\mathbb{N}} J_i \;\Rightarrow\; \sum_{n\in\mathbb{N}} \mathrm{length}(I_n) \leqslant \sum_{i\in\mathbb{N}} \mathrm{length}(J_i) \tag{38.20}$$

A proof of this uses arguments from measure theory (see Subsection 38.2 below) which we do not want to go into here.

Moving to the actual proof, for each $n\in\mathbb{N}$, let $\{J_{n,k}\}_{k=0}^{r_n}$ be open intervals such that (38.17) hold. Denote

$$O_n := \bigcup_{m\geqslant n}\bigcup_{k=0}^{r_n} J_{n,k} \tag{38.21}$$

Then O_n is an open subset of (a,b) and so there exists a family $\{I_{n,i}\}_{i\in\mathbb{N}}$ of disjoint open (or empty) subintervals of (a,b) such that $O_n = \bigcup_{i\in\mathbb{N}} I_{n,i}$. Using (38.20) we then get

$$b-a \geqslant \sum_{i\in\mathbb{N}} \mathrm{length}(I_{n,i}) \geqslant \sum_{k=0}^{r_n} \mathrm{length}(J_{n,k}) \geqslant \eta'. \tag{38.22}$$

The fact that the first sum is finite implies the existence of $i_n\in\mathbb{N}$ such that

$$\sum_{i>i_n} \mathrm{length}(I_{n,i}) \leqslant 2^{-n-3}\eta' \tag{38.23}$$

Now let $I'_{n,i}\subseteq I_{n,i}$ be a closed interval centered at the same point as $I_{n,k}$ and length

$$\mathrm{length}(I'_{n,i}) = \mathrm{length}(I_{n,i}) - 2^{-n-i-4}\eta' \tag{38.24}$$

whenever the right-hand side is positive and $I_{n,i'}=\emptyset$ otherwise. Define

$$C_n := \bigcap_{m=0}^{n}\bigcup_{i=0}^{i_m} I'_{m,i} \tag{38.25}$$

Being the intersection of finite unions of closed intervals, C_n is closed and $C_{n+1}\subseteq C_n$ holds for each $n\in\mathbb{N}$.

Next observe that

$$O_n \smallsetminus C_n = \bigcup_{m=0}^{n}\Biggl(\Bigl(\bigcup_{i>i_m} I_{m,i}\Bigr)\cup\Bigl(\bigcup_{i=0}^{i_m}(I_{m,i}\smallsetminus I'_{m,i})\Bigr)\Biggr) \tag{38.26}$$

The total length of the intervals on the right is bounded by

$$\begin{aligned}\sum_{m=0}^{n}\sum_{i>i_m}\mathrm{length}(I_{m,i}) + \sum_{m=0}^{n}\sum_{i=0}^{i_m}\bigl[\mathrm{length}(I_{m,i}) - \mathrm{length}(I'_{m,i})\bigr]&\\ \leqslant \sum_{m=0}^{n} 2^{-m-3}\eta' + \sum_{m=0}^{n} 2^{-m-4}\sum_{i=0}^{i_n}\eta' 2^{-i} \leqslant \frac{1}{2}\eta'&\end{aligned} \tag{38.27}$$

Since both O_n and $O_n\smallsetminus C_n$ are countable unions of open intervals, $C_n=\emptyset$ with the help of (38.20) and (38.21–38.22) would force the total length of the intervals constituting $O_n\smallsetminus C_n$ to be at least η'. From (38.27) we thus conclude $C_n\neq\emptyset$ for all $n\in\mathbb{N}$.

Since $\{C_n\}_{n\in\mathbb{N}}$ are non-empty, closed and nested subsets of $[a,b]$, the Cantor intersection property implies $\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$. But $C_n \subseteq O_n$ for each $n \in \mathbb{N}$ and so we also have $\bigcap_{n\in\mathbb{N}} O_n \neq \emptyset$, which is the desired claim. □

One of the basic applications of Osgood's theorem is the continuity of integrals with respect to an underlying parameter:

Corollary 38.4 *Let $I \subseteq \mathbb{R}$ be an open interval and $f\colon [a,b]\times I \to \mathbb{R}$ such that*

(1) *for all $x \in [a,b]$, the function $\alpha \mapsto f(x,\alpha)$ is continuous, and*
(2) *for all $\alpha \in I$, the function $x \mapsto f(x,\alpha)$ is Riemann integrable.*

Assume in addition that

$$\sup_{\alpha\in I}\sup_{x\in[a,b]}\bigl|f(x,\alpha)\bigr| < \infty \tag{38.28}$$

Then $h\colon I \to \mathbb{R}$ defined by $h(\alpha) := \int_a^b f(x,\alpha)\mathrm{d}x$ is continuous on I.

Proof. Let $\alpha \in I$ and $\{\alpha_n\}_{n\in\mathbb{N}} \in I^{\mathbb{N}}$ be such that $\alpha_n \to \alpha$. Set $g_n := f(\cdot,\alpha_n)$ and $g := f(\cdot,\alpha)$. Then $h_n \to h$ pointwise by (1) with g_n and h Riemann integrable by (2). As the functions are also bounded, Theorem 38.2 implies $h(\alpha_n) \to h(\alpha)$. By sequential characterization of continuity, this implies that h is continuous at α. □

A similar theorem can then be derived for differentiation of the integral with respect to this parameter. However, as this involves functions of more than one variable, this will be treated only if and when we get to multivariate calculus.

38.2 Lebesgue integral.

These aforementioned shortcomings of Riemann's theory provided the important motivation for the creation of more advanced theory of *Lebesgue integral*. The main novelty, going perhaps back to the *Cavalieri principle*, is that instead of partitioning the domain of f, we partition the range of f — say, into intervals of the form $\{[\frac{k}{n},\frac{k+1}{n})\colon k \in \mathbb{Z}\}$ — and use the preimage map to approximate the "area under the graph of f" by

$$\sum_{k\in\mathbb{Z}} \frac{k}{n}\,\text{length}\Bigl(f^{-1}\bigl([\tfrac{k}{n},\tfrac{k+1}{n})\bigr)\Bigr) \tag{38.29}$$

instead of the Riemann sums. (Note that the sum is effectively finite as soon as f is bounded.) However, this "minor" change requires us to first develop a robust theory of "length" for sets that are more complicated than just intervals or finite unions thereof. The resulting *measure theory* is the subject of early graduate Analysis courses.

We have already encountered some aspects of measure theory above and also in the discussion of sets of zero length. Indeed, a natural way to extend the notion of the length to all subsets of $\mathbb{R}$ via

$$\lambda^\star(A) := \inf\left\{\sum_{i=0}^{\infty}\text{length}(I_n)\colon \{I_n\}_{n\in\mathbb{N}} \text{ open intervals } \wedge A \subseteq \bigcup_{n\in\mathbb{N}} I_n\right\} \tag{38.30}$$

(Calling A zero length just meant $\lambda^\star(A) = 0$.) Unfortunately, the map $\lambda^\star\colon \mathcal{P}(A) \to [0,\infty]$ lacks a very reasonable property which is factorization under complements. Namely, for

$A \subseteq B$ bounded, we would like to have

$$\lambda^\star(A) + \lambda^\star(B \smallsetminus A) = \lambda^\star(B) \tag{38.31}$$

but this fails in general because, roughly speaking, the boundary of A may be counted twice on the left-hand side. A solution is to identify a suitable subset of $\mathcal{P}(\mathbb{R})$ for which this factorization holds. The resulting "special" sets are then called *measurable*.

As it turns out, the collection of measurable sets is closed under countable unions and complements. Moreover, the function $\lambda^\star$ acts additively on countable unions of disjoint measurable sets which makes the restriction of $\lambda^\star$ to the measurable sets a *measure*. Functions that preimage intervals into measurable sets are called *measurable functions*; as it turns out, if a measurable function is also bounded then the sums in (38.29) converge as $n \to \infty$ and define the Lebesgue integral. However, boundedness is not required: measurable functions for which the limit exists are called *Lebesgue integrable*.

The measure theoretic approach to integration turns out to be extremely versatile allowing us to integrate real-valued functions over far more complicated sets than just the reals; e.g., over subsets of $\mathbb{R}^d$, spaces of functions, linear operators, graphs, metric spaces, etc. In spite of this generality, even that theory is not void of shortcomings that we saw for the Riemann integral. Indeed, while handling quite properly pointwise convergence, the Fundamental Theorem of Calculus, part II, still fails to hold without additional provisos — namely, the derivative has to be absolutely integrable. (This integrals in (38.3) and (38.4) thus do not exist in Lebesgue theory either.)

Other approaches to integration have therefore been considered, e.g., by Saks, Perron, Ward, and Lusin culminating in the work of J. Kurzweil and, independently, R. Henstock that we will discuss next.

38.3 Henstock-Kurzweil integral.

Suppose we put ourselves to the task of extending the Riemann integral to a larger class of functions while preserving the framework that extracts the integral as a limit of sorts of Riemann sums over marked partitions. In order to allow for unbounded functions, we clearly cannot work with arbitrary marked points but have to allow only those that lie in partition intervals of small-enough size. This leads to the following concept:

Definition 38.5 (Gauge function and associated partitions) *Let $a < b$ be reals. Given function $\gamma\colon [a,b] \to (0,\infty)$, referred to as* gauge function *in this context, a marked partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ of $[a,b]$ is said to be γ-*fine *if*

$$\forall i = 1,\dots,n\colon [t_{i-1}, t_i] \subseteq \bigl[t_i^\star - \gamma(t_i^\star), t_i^\star + \gamma(t_i^\star)\bigr]. \tag{38.32}$$

We will now examine the Riemann sums over partitions that are γ-fine for a given gauge function γ. However, for this we need to know that each gauge function admits at least one marked partition of this kind. This will follow from:

Theorem 38.6 (P. Cousin, 1895) *Let $a < b$ be reals and let that $\mathcal{I}$ be a collection of non-degenerate closed subintervals of $[a,b]$ with the following property: For each $x \in [a,b]$ there is $\delta > 0$ such that all non-degenerate closed intervals $[c,d]$ satisfying*

$$[c,d] \subseteq [a,b] \ \wedge\ x \in [c,d] \ \wedge\ d - c < \delta \tag{38.33}$$

belong to $\mathcal{I}$. Then there is a partition of $[a,b]$ consisting only of intervals in $\mathcal{I}$, i.e., that there are $a = t_0 < t_1 < \dots < t_n = b$ satisfying

$$\forall i = 1,\dots,n\colon\ [t_{i-1},t_i] \in \mathcal{I} \tag{38.34}$$

The proof of Cousin's theorem is left to a homework exercise. Given a gauge function $\gamma\colon [a,b] \to (0,\infty)$, the set

$$\mathcal{I} := \bigcup_{t\in[a,b]} \bigl\{[c,d] \cap [a,b] \colon t-\gamma(t) \leqslant c < d \leqslant t+\gamma(t)\bigr\} \tag{38.35}$$

contains all intervals that can possibly appear in γ-fine partitions of $[a,b]$. That one such partition actually exists is a restatement of (38.34).

We remark that Cousin's motivation for this result was to extend the Heine-Borel theorem from countable covers to arbitrary covers. However, as the wiki page on this result points out: "...Pierre Cousin did not receive any credit. Cousin's theorem was generally attributed to H. Lebesgue as the Borel-Lebesgue theorem. Lebesgue was aware of this result in 1898, and proved it in his 1903 dissertation."

Returning to our main line of thought, we now put forward:

Definition 38.7 (Henstock-Kurzweil integral) *A function $f\colon [a,b] \to \mathbb{R}$ is said to be* Henstock-Kurzweil integrable *if*

$$\exists L \in \mathbb{R}\ \forall \epsilon > 0\ \exists \gamma \in (0,\infty)^{[a,b]}\ \forall \Pi = \gamma\text{-fine partition}\colon \bigl|R(f,\Pi) - L\bigr| < \epsilon. \tag{38.36}$$

The quantity L, which is unique if exists at all, is the Henstock-Kurzweil integral.

Note that for a constant gauge function, the Henstock-Kurzweil integral degenerates to the Riemann integral and so

$$f \text{ Riemann integrable on } [a,b] \ \Rightarrow\ f \text{ Henstock-Kurzweil integrable on } [a,b] \tag{38.37}$$

But there are many functions for which the reverse implication fails. The most elementary example is that of the Dirichlet function $1_{\mathbb{Q}}$ which we have shown is not Riemann integrable. Yet we have:

Lemma 38.8 *For all $a < b$ real, the Dirichlet function $1_{\mathbb{Q}}$ is Henstock-Kurzweil interable on $[a,b]$ and $\int_a^b 1_{\mathbb{Q}}(x)\mathrm{d}x = 0$.*

Proof. Let $a < b$. Fix $\epsilon > 0$ and let $\{q_n\}_{n\in\mathbb{N}}$ enumerate all rationals in $[a,b]$. Define

$$\gamma(t) := \begin{cases} \epsilon 2^{-n}, & \text{if } t = q_n \text{ for some } n \in \mathbb{N}, \\ 1, & \text{if } t \in [a,b] \smallsetminus \mathbb{Q}. \end{cases} \tag{38.38}$$

Then for any γ-fine partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ of $[a,b]$,

$$\bigl|R(f,\Pi)\bigr| \leqslant \sum_{i=1}^n 1_{\mathbb{Q}}(t_i^\star)|t_i - t_{i-1}| \leqslant \sum_{i=1}^n 1_{\mathbb{Q}}(t_i^\star) 2\gamma(t_i^\star) \leqslant \sum_{n=0}^\infty \epsilon 2^{-n+1} = 4\epsilon \tag{38.39}$$

where the first inequality is the triangle inequality for the absolute value, the second inequality follows from (38.32) and the third inequality from the definition of γ. This proves that (38.36) holds with $L := 0$. $\square$

In order to demonstrate the strength of the Henstock-Kurzweil integration theory, we now show that the second Fundamental Theorem of Calculus holds in this theory with no provisos on the derivative:

Theorem 38.9 (FTCII for Henstock-Kurzweil integral) *Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Choosing $f'(a)$ and $f'(b)$ arbitrarily, f' is then Henstock-Kurzweil integrable on $[a,b]$ and*

$$\int_a^b f'(x)\mathrm{d}x = f(b) - f(a) \tag{38.40}$$

Proof. For each $t \in (a,b)$, the existence of $f'(t)$ shows that

$$\gamma(t) := \frac{1}{2}\sup\left\{\delta \in (0,1)\colon \sup_{\substack{x\in[a,b]\\ 0<|x-t|<\delta}} \left|\frac{f(x)-f(t)}{x-t} - f'(t)\right| < \frac{\epsilon}{b-a}\right\} \tag{38.41}$$

is positive and finite. Choosing $f'(a)$ and $f'(b)$ arbitrarily, set also

$$\gamma(a) := \min\left\{\frac{1}{b-a}\,\frac{\epsilon}{1+|f'(a)|}, \frac{1}{2}\sup\left\{\delta \in (b-a)\colon \sup_{x\in[a,a+\delta]}\bigl|f(x)-f(a)\bigr| < \frac{\epsilon}{b-a}\right\}\right\} \tag{38.42}$$

and similarly

$$\gamma(b) := \min\left\{\frac{1}{b-a}\,\frac{\epsilon}{1+|f'(a)|}, \frac{1}{2}\sup\left\{\delta \in (b-a)\colon \sup_{x\in[b-\delta,b]}\bigl|f(x)-f(a)\bigr| < \frac{\epsilon}{b-a}\right\}\right\} \tag{38.43}$$

for each $t,x \in [a,b]$ we then have

$$x \in \bigl[t-\gamma(t), t+\gamma(t)\bigr] \ \Rightarrow\ \bigl|f(x)-f(t)-f'(t)(t-x)\bigr| < \frac{2\epsilon}{b-a}|x-t| \tag{38.44}$$

Let $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ be a γ-fine partition of $[a,b]$. Then the same calculation as in the proof of Theorem 35.7 shows

$$\begin{aligned}
\bigl|f(b)-f(a)-R(f',\Pi)\bigr| &\leqslant \sum_{i=1}^n \bigl|f(t_i)-f(t_{i-1})-f'(t_i^\star)(t_i-t_{i-1})\bigr| \\
&\leqslant \sum_{i=1}^n \bigl|f(t_i)-f(t_i^\star)-f'(t_i^\star)(t_i-t_i^\star)\bigr| + \sum_{i=1}^n \bigl|f(t_{i-1})-f(t_i^\star)-f'(t_i^\star)(t_{i-1}-t_i^\star)\bigr| \\
&\leqslant \sum_{i=1}^n \frac{2\epsilon}{b-a}(t_i-t_i^\star) + \sum_{i=1}^n \frac{2\epsilon}{b-a}(t_i^\star-t_{i-1}) = \frac{4\epsilon}{b-a}\sum_{i=1}^n (t_i-t_{i-1}) = 4\epsilon
\end{aligned} \tag{38.45}$$

It follows that f' is Henstock-Kurzweil integrable with $\int_a^b f'(x)\mathrm{d}x = f(b)-f(a)$. □

Every Henstock-Kurzweil integrable function is measurable, but the Henstock-Kurzweil integral is actually slightly more general than the Lebesgue integral. Indeed:

Theorem 38.10 *A function $f\colon [a,b] \to \mathbb{R}$ is Lebesgue integrable on $[a,b]$ if and only if f and $|f|$ are Henstock-Kurzweil integrable.*

The function $x \mapsto \frac{\sin(1/x)}{x}$ treated in (38.3) using the improper Riemann integral is (properly) Henstock-Kurzweil integrable yet (not being absolutely integrable) it is not Lebesgue integrable. This is because in Lebesgue's theory of integration, a measurable f is integrable if and only if $|f|$ is integrable. That being said, the Lebesgue theory is superior in its generality of the underlying space which is why it is the dominant integration theory used throughout mathematics.