## 37. CONDITIONS FOR STIELTJES INTEGRABILITY

The aim of this section is to give reasonable sufficient conditions for Stieltjes integrability. However, we start by discussing some basic operations with the integral that are useful in the overall goal of this section as well.

### 37.1 Integration by parts and substitution rule.

The Stieltjes integral retains (or extends) various properties valid for the Riemann integral; namely, the Integration by parts from Corollary 35.9 and the Substitution rule from Corollary 35.10. The proof of these for the Riemann integral relied on the Fundamental Theorem of Calculus which for the Stieltjes integral takes the form:

**Lemma 37.1** (Reduction to Riemann integral) *Let $f,g\colon [a,b]\to\mathbb{R}$ be such that*

(1) *$f$ is Riemann integrable on $[a,b]$, and*
(2) *$g$ is continuous on $[a,b]$, differentiable on $(a,b)$ with $g'$ Riemann integrable on $[a,b]$.*

*Then $f$ is Stieltjes integrable with respect to $g$ on $[a,b]$ and*

$$\int_a^b f\mathrm{d}g = \int_a^b f(x)g'(x)\mathrm{d}x \tag{37.1}$$

*Proof.* The proof is very similar to the proof of Theorem 35.7 and so we omit it. □

Note that Lemma 37.1 subsumes the FTCII thanks to the fact that $\int_a^b 1\mathrm{d}g = g(b)-g(a)$, as follows from $S(1,\mathrm{d}g,\Pi) = g(b)-g(a)$ for any partition $\Pi$. While the reduction to the Riemann integral is often used to evaluate the Stieltjes integral, the Integration by Parts and Substitution rule we will state and prove next do NOT rely on this reduction and are thus more general. Indeed, we have:

**Lemma 37.2** (Integration by parts) *For all $f,g\colon [a,b]\to\mathbb{R}$*

$$f\in \mathrm{RS}(g,[a,b]) \;\Leftrightarrow\; g\in\mathrm{RS}(f,[a,b]) \tag{37.2}$$

*and, if both TRUE, then*

$$\int_a^b f\mathrm{d}g = f(b)g(b)-f(a)g(a)-\int_a^b g\,\mathrm{d}f \tag{37.3}$$

*Proof.* The key point of the proof is that, for any marked partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ of $[a,b]$, the pair $\Pi' = (\{t_i^\star\}_{i=0}^{n+1}, \{t_{i-1}\}_{i=1}^{n+1})$, where $t_0^\star := a$ and $t_{n+1}^\star := b$, is another marked partition of $[a,b]$. A calculation shows

$$\begin{aligned}
f(a)g(a)+S(f,\mathrm{d}g,\Pi) &= f(a)g(a)+\sum_{i=1}^n f(t_i^\star)\bigl(g(t_i)-g(t_{i-1})\bigr)\\
&= \sum_{i=0}^n f(t_i^\star)g(t_i) - \sum_{i=0}^n f(t_{i+1}^\star)g(t_i)+f(b)g(b)\\
&= \sum_{i=0}^n g(t_i)\bigl[f(t_i^\star)-f(t_{i+1}^\star)\bigr]+f(b)g(b)\\
&= -S(g,\mathrm{d}f,\Pi')+f(b)g(b)
\end{aligned} \tag{37.4}$$

Now assume that $g \in \mathrm{RS}(f,[a,b])$ and pick $\epsilon > 0$. Then $|S(g,\mathrm{d}f,\Pi') - \int_a^b g\mathrm{d}f| < \epsilon$ as soon as $\|\Pi'\| < \delta$, for $\delta$ related to $\epsilon$ as in Definition 31.2. But $\|\Pi'\| \leqslant 2\|\Pi\|$ and so if $\|\Pi\| < \delta/2$, then (37.4) shows that $S(f,\mathrm{d}g,\Pi)$ is within $\epsilon$ of the right-hand side of (37.3). It follows that $f \in \mathrm{RS}(g,[a,b])$, proving $\Leftarrow$ in (37.2), and that the identity (37.3) holds. (The equivalence in (37.2) holds by symmetry.) ☐

Note that writing $\mathrm{d}g = g'(x)\mathrm{d}x$ and $\mathrm{d}f = f'(x)\mathrm{d}f$, the previous lemma subsumes the statement of Corollary 35.9. Note that combining Lemmas 37.2 and 37.2 we get:

**Corollary 37.3** *Suppose $f,g\colon [a,b] \to \infty$ are such that $f$ is Riemann integrable and $g$ is continuous on $[a,b]$. Moreover, assume that $g$ is differentiable on $[a,b]$ with $g'$ Riemann integrable. Then $g \in \mathrm{RS}(f,[a,b])$ and*

$$\int_a^b g\,\mathrm{d}f = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\mathrm{d}x \tag{37.5}$$

Note that this again amounts to the use of the formal expression $\mathrm{d}g = g'(x)\mathrm{d}x$. Moving to the Substitution rule, here we get:

**Lemma 37.4** (Substitution) *Let $g,h\colon [a,b] \to \mathbb{R}$ and, assuming $g \in \mathrm{RS}(h,[a,b])$, let*

$$\forall x \in [a,b]\colon\ \ G(x) := \int_a^x g\,\mathrm{d}h \tag{37.6}$$

*Then for all bounded $f\colon [a,b] \to \mathbb{R}$,*

$$f \in \mathrm{RS}(G,[a,b]) \ \Leftrightarrow\ f\cdot g \in \mathrm{RS}(h,[a,b]) \tag{37.7}$$

*and, assuming that both sides are TRUE,*

$$\int_a^b f\mathrm{d}G = \int_a^b f\cdot g\,\mathrm{d}h \tag{37.8}$$

*Here $(f\cdot g)(x) := f(x)g(x)$.*

*Proof.* Assume $g \in \mathrm{RS}(h,[a,b])$. The key point of the proof is the following approximation claim: For each $\epsilon > 0$ there is $\delta > 0$ for any marked partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ with $\|\Pi\| < \delta$, we have

$$\sum_{i=1}^n \left| g(t_i^\star)\bigl[h(t_i) - h(t_{i-1})\bigr] - \int_{t_{i-1}}^{t_i} g\mathrm{d}h \right| < \epsilon \tag{37.9}$$

To see why this is true, let $\delta > 0$ be related to $\epsilon > 0$ as in the definition of Stieltjes integrability. Given a partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ with $\|\Pi\| < \delta$, let

$$I := \left\{ i = 1,\dots,n\colon \int_{t_{i-1}}^{t_i} g\mathrm{d}h > g(t_i^\star)\bigl[h(t_i) - h(t_{i-1})\bigr] \right\} \tag{37.10}$$

Since $g \in \mathrm{RS}(h,[t_{i-1},t_i])$, for each $i \in I$ there is a partition $\Pi_i$ of $[t_{i-1},t_i]$ such that

$$\left| S(g,\mathrm{d}h,\Pi_i) - \int_{t_{i-1}}^{t_i} g\mathrm{d}h \right| < \frac{\epsilon}{n} \tag{37.11}$$

Now consider the partition $\widetilde{\Pi}$ that contains all partition points of the partitions $\Pi$ and $\Pi_i$ for all $i \in I$, and the marked points of $\Pi$ in intervals indexed by $i \notin I$ and all the marked points of the partitions $\Pi_i$ with $i \in I$. The additivity of the Stieltjes sum and the integral then shows

$$S(f, \mathrm{d}g, \widetilde{\Pi}) - \int_a^b g\mathrm{d}h = \sum_{i=1}^n \biggl( g(t_i^\star)\bigl[h(t_i) - h(t_{i-1})\bigr] - \int_{t_{i-1}}^{t_i} g\mathrm{d}h \biggr)^+ + \sum_{i\in I}\Bigl( S(g, \mathrm{d}h, \Pi_i) - \int_{t_{i-1}}^{t_i} g\mathrm{d}h \Bigr) \tag{37.12}$$

where the use of the positive part $a^+ := \max\{0, a\}$ effectively eliminates terms with $i \in I$ from the first sum. Since $\|\widetilde{\Pi}\| < \delta$, the the left-hand side is at least $-\epsilon$. The fact that (37.11) holds for each $\Pi_i$ with $i \in I$ in turn ensures that the second sum on the right is at most $\epsilon$. Hence we get

$$\sum_{i=1}^n \biggl( g(t_i^\star)\bigl[h(t_i) - h(t_{i-1})\bigr] - \int_{t_{i-1}}^{t_i} g\mathrm{d}h \biggr)^+ < 2\epsilon \tag{37.13}$$

Since the same applies to the sum of the negative parts, we get (37.9) with $4\epsilon$ instead of $\epsilon$ on the right-hand side.

Using (37.9) we now quickly finish the claim. Let $f\colon [a,b] \to \mathbb{R}$ and let $\Pi$ be a partition of $[a,b]$. Then (37.6) and additivity of the integral give

$$S(f\cdot g, \mathrm{d}h, \Pi) - S(f, \mathrm{d}G, \Pi) = \sum_{i=1}^n f(t_i^\star)\biggl[ g(t_i^\star)\bigl[h(t_i) - h(t_{i-1})\bigr] - \int_{t_{i-1}}^{t_i} g\mathrm{d}h \biggr] \tag{37.14}$$

Assuming that $f$ is bounded, the right-hand side is bounded by $\|f\|$ times the quantity in (37.9). So the convergence of $S(f, \mathrm{d}g, \Pi)$ as $\|\Pi\| \to 0$ is equivalent to the convergence of $S(f, \cdot g, \mathrm{d}h, \Pi)$ and both "limits" (if they exist) are equal. □

As it turns out, the conclusion of Lemma 37.4 may fail when $f$ is unbounded. Indeed, set $a := 0$, $b := 1$, $g(x) = h(x) := x$ and $f(x) := 1/x$ for $x > 0$ and $f(0) = 0$. Then $G(x) = \frac{1}{2}x^2$ and $f \notin \mathrm{RS}(G, [0,1])$ because $f$ is unbounded on intervals on which $G$ is non-constant. Yet $f \cdot g = 1$ and so $f \cdot g \in \mathrm{RS}(h, [0,1])$. This is clearly because the Stieltjes integral works only with finite partitions which does not allow us to refine the intervals near zero so that $f(t_i^\star)(G(t_i) - G(t_{i-1}))$ is summable. This is an aspect that is fixed in more advanced theories of integration that we will mention in Section 38.

### 37.2 Conditions for Stieltjes integrability.

In our discussion of the Stieltjes integral, we have so far given only one necessary condition for integrability (namely, (36.13) in Lemma 36.6). Another necessary condition was proved along with Lemma 37.4:

**Corollary 37.5** *Let $f, g\colon [a,b] \to \mathbb{R}$ be such that $f \in \mathrm{RS}(g, [a,b])$. For each $\epsilon > 0$ there is $\delta > 0$ such that if $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ is a marked partition of $[a,b]$ with $\|\Pi\| < \delta$, then*

$$\sum_{i=1}^n \biggl| f(t_i^\star)\bigl[g(t_i) - g(t_{i-1})\bigr] - \int_{t_{i-1}}^{t_i} f\mathrm{d}g \biggr| < \epsilon \tag{37.15}$$

*Proof.* This follows from (37.13) and a corresponding statement for the negative part. □

Concerning sufficient conditions for integrability, we had one based on conversion to the Riemann integral (Lemma 37.1). A useful necessary and sufficient condition is:

**Lemma 37.6** (Cauchy criterion for Stieltjes integral) *Let $f, g\colon [a,b] \to \mathbb{R}$. Then $f \in \text{RS}(g,[a,b])$ (and equivalently $g \in \text{RS}(f,[a,b])$) if and only if for each $\epsilon > 0$ there is a $\delta > 0$ such that for all marked partitions $\Pi, \Pi'$ of $[a,b]$,*

$$\max\bigl\{\|\Pi\|, \|\Pi'\|\bigr\} < \delta \;\Rightarrow\; \bigl|S(f,\mathrm{d}g,\Pi) - S(f,\mathrm{d}g,\Pi')\bigr| < \epsilon. \tag{37.16}$$

*Proof.* This is a direct consequence of the definition of Stietljes integrability. □

A somewhat deeper sufficient criterion concerns Stieltjes integrals with respect to functions of bounded variation:

**Lemma 37.7** *Let $f, g\colon [a,b] \to \mathbb{R}$ be such that $f$ is bounded and $g$ is of bounded variation, i.e., $V(g,[a,b]) < \infty$, and such that $f$ and $g$ have no common discontinuity points. If for each $\epsilon > 0$ there is a partition $\Pi = \{t_i\}_{i=0}^n$ of $[a,b]$ such that*

$$\sum_{i=1}^n \text{osc}\bigl(f,[t_{i-1},t_i]\bigr)V\bigl(g,[t_{i-1},t_i]\bigr) < \epsilon \tag{37.17}$$

*then $f \in \text{RS}(g,[a,b])$.*

*Proof.* An earlier version of this statement did not have the requirement of no common discontinuity points and referred the proof to that of Theorem 32.9. However, this would prove integrability in the Darboux-Stieltjes sense (and also the Moore-Pollard sense in Definition 36.7) which works with only the requirement of no common same-sided discontinuities, rather than Riemann-Stieltjes sense. We thus have to work a bit harder.

We start by proving that, for $f$ bounded and $g$ bounded variation with no common discontinuity points,

$$\forall\epsilon > 0\,\exists\delta > 0\,\forall s,t \in [a,b]\colon\; 0 < t - s < \delta \;\Rightarrow\; \text{osc}\bigl(f,[s,t]\bigr)V\bigl(g,[s,t]\bigr) < \epsilon. \tag{37.18}$$

Extend $f$ by $f(a)$ to the left of $a$ and by $f(b)$ to the right of $b$, and similarly for $g$. Consider the dyadic intervals $I_{n,k} := [(k-1)2^{-n}, k2^n]$ indexed by $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Given $\epsilon > 0$, denote

$$C_n := \bigcup\bigl\{I_{n,k}\colon \text{osc}(f,I_{n,k})V(g,I_{n,k}) \geqslant \epsilon\bigr\} \tag{37.19}$$

Suppose $\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$ and let $x \in \bigcap_{n\in\mathbb{N}} C_n$. We claim that neither $f$ nor $g$ is then continuous at $x$. Indeed, for each $n \in \mathbb{N}$, let $k_n := \min\{k \in \{0,\dots,2^n-1\}\colon x \in I_{n,k}\}$. Abbreviate $J_n := I_{n,k_n}$ and note that then $\forall n \in \mathbb{N}\colon \text{osc}(f,J_n)V(g,J_n) \geqslant \epsilon$. This implies $V(g,J_n) \geqslant \epsilon/[1+\sup_{t\in[a,b]}|f(t)|]$ showing that $t \mapsto V(g,[a,t])$, and thus also $t \mapsto g(t)$ is NOT continuous at $x$. Similarly we get $\text{osc}(f,[x-\delta,x+\delta]) \geqslant \epsilon/[1+V(g,[a,b])]$ and so (34.9) shows that $f$ is NOT continuous at $x$.

It follows that no such $x$ can exist and so we have $\bigcap_{n\in\mathbb{N}} C_n = \emptyset$. As $I \subseteq J$ implies $\text{osc}(f,I)V(g,I) \subseteq \text{osc}(f,J)V(g,J)$ and each $I_{n+1,k}$ is contained in one $I_{n,j}$, the sets $\{C_n\}_{n\in\mathbb{N}}$ are nested, i.e., $\forall n \in \mathbb{N}\colon C_{n+1} \subseteq C_n$. As these sets are also closed and, being subsets of $[a-1,b+1]$, thus compact, the assumption $\bigcap_{n\in\mathbb{N}} C_n = \emptyset$ forces $\exists n \in \mathbb{N}\colon C_n = \emptyset$, by the Cantor Intersection Property. Given $n$ with $C_n \neq \emptyset$, any closed interval $J$ contained in one of $I_{n,k}$ obeys $\text{osc}(f,J)V(g,J) < \epsilon$. Performing the same argument with $I_{n,k}$ replaced by $[(k-1/2)2^{-n},(k+1/2)2^{-n}]$ shows $\operatorname{osc}(f,J)V(g,J) < \epsilon$ for all closed intervals $J$ of length less than $2^{-n-1}$, proving (37.18) with $\delta := 2^{-n-1}$.

We now move to the proof of $f \in \mathrm{RS}(g,[a,b])$. Given $\epsilon > 0$, let $\Pi := \{t_i\}_{i=1}^n$ be a partition satisfying (37.17). By (37.18), there exists $\delta > 0$ such that $0 < t-s < \delta$ implies $\operatorname{osc}(f,[s,t])V(g,[s,t]) < \epsilon/n$. Let $\Pi' = (\{t_i'\}_{i=1}^m, \{t_i^\star\}_{i=1}^m)$ be a marked partition with $\|\Pi'\| < \delta$ and let $\Pi''$ be the refinement of $\Pi'$ by points of $\Pi$ using the marked points of $\Pi'$ in the intervals they fall into and right-endpoints of intervals that do not contain a marked point of $\Pi'$. Let $K$ be the set of those $i \in \{1,\dots,n\}$ such that $[t_{i-1}', t_i']$ contains a partition point of $\Pi$, write $r_i$ for the total number intervals that these partition points split $[t_{i-1}', t_i']$ into and index the right-endpoints of these intervals by $\{s_{i,j}\}_{j=1}^{r_i}$. Then

$$S(f,\mathrm{d}g,\Pi'') - S(f,\mathrm{d}g,\Pi') = \sum_{i\in K}\sum_{j=1}^{r_i}\bigl[f(s_{i,j}) - f(t_i^\star)\bigr]\bigl[g(t_{i,j}) - g(t_{i,j-1})\bigr] \tag{37.20}$$

and so

$$\begin{aligned}\bigl|S(f,\mathrm{d}g,\Pi'') - S(f,\mathrm{d}g,\Pi')\bigr| &\leqslant \sum_{i\in K}\sum_{j=1}^{r_i}\bigl|f(s_{i,j}) - f(t_i^\star)\bigr|\bigl|g(t_{i,j}) - g(t_{i,j-1})\bigr| \\ &\leqslant \sum_{i\in K}\operatorname{osc}\bigl(f,[t_{i-1}',t_i']\bigr)\sum_{j=1}^{r_i}\bigl|g(t_{i,j}) - g(t_{i,j-1})\bigr| \\ &\leqslant \sum_{i\in K}\operatorname{osc}\bigl(f,[t_{i-1}',t_i']\bigr)V\bigl(g,[t_{i-1}',t_i']\bigr) < \epsilon,\end{aligned} \tag{37.21}$$

where the last inequality comes from our choice of $\delta$ and the fact that $|K| \leqslant n$.

Endowing $\Pi$ with left-endpoint marked points, a completely analogous derivation (using that $\Pi \subseteq \Pi''$) in turn shows

$$\bigl|S(f,\mathrm{d}g,\Pi'') - S(f,\mathrm{d}g,\Pi)\bigr| \leqslant \sum_{i=1}^n \operatorname{osc}\bigl(f,[t_{i-1},t_i]\bigr)V\bigl(g,[t_{i-1},t_i]\bigr) \overset{(37.17)}{<} \epsilon \tag{37.22}$$

and so we get that, for all marked partitions $\Pi'$,

$$\|\Pi'\| < \delta \;\Rightarrow\; \bigl|S(f,\mathrm{d}g,\Pi') - S(f,\mathrm{d}g,\Pi)\bigr| < 2\epsilon. \tag{37.23}$$

Using the Cauchy criterion (Lemma 37.6), it follows that $f \in \mathrm{RS}(g,[a,b])$. □

Hereby we conclude:

**Corollary 37.8** *Let $f,g\colon [a,b] \to \mathbb{R}$ be such that $f$ is continuous on $[a,b]$ and $g$ is of bounded variation on $[a,b]$. Then $f \in \mathrm{RS}(g,[a,b])$ as well as $g \in \mathrm{RS}(f,[a,b])$ and*

$$\left|\int_a^b f\mathrm{d}g\right| \leqslant \Bigl(\sup_{x\in[a,b]}\bigl|f(x)\bigr|\Bigr)V\bigl(g,[a,b]\bigr) \tag{37.24}$$

*Proof.* The integrability of $f$ with respect to $g$ is proved using the bound Lemma 37.7 combined with the fact that, by uniform continuity of $f$, for each $\epsilon > 0$ there is $\delta > 0$ such that $\operatorname{osc}(f,[s,t]) < \epsilon/(b-a)$ whenever $t-s < \delta$. The integrability of $g$ with respect to $f$ then follows from Lemma 37.2. The bound on the integral inherited from the corresponding bound on $S(f,\mathrm{d}g,\Pi)$. □

Note that the additivity of the integral then implies that, for $f, \tilde f\colon [a,b]\to\mathbb{R}$ continuous and $g,\tilde g\colon [a,b]\to\mathbb{R}$ bounded variation,

$$\left|\int_a^b f\mathrm{d}g-\int_a^b \tilde f\mathrm{d}\tilde g\right|\leqslant \big(\sup|f-\tilde f|\big)V\big(g,[a,b]\big)+\big(\sup|f|\big)V\big(g-\tilde g,[a,b]\big) \tag{37.25}$$

This is a statement of continuity of $f\mapsto\int_a^b f\mathrm{d}g$ in the supremum norm $f\mapsto\sup|f|$ and the continuity of $g\mapsto\int_a^b f\mathrm{d}g$ in the variational norm $g\mapsto V(g,[a,b])$.

**Lemma 37.9** *Let $f,g\colon[a,b]\to\mathbb{R}$ be functions and assume that $f$ is bounded and $g$ is of bounded variation; i.e., $V(g,[a,b])<\infty$. Let $v_g\colon[a,b]\to\mathbb{R}$ be defined by $v_g(t):=V(g,[a,t])$. Then*

$$f\in \mathrm{RS}\big(g,[a,b]\big)\;\Leftrightarrow\; f\in\mathrm{RS}\big(v_g,[a,b]\big) \tag{37.26}$$

*and, if both TRUE, then also $|f|\in\mathrm{RS}(v_g,[a,b])$ and*

$$\left|\int_a^b f\mathrm{d}g\right|\leqslant\int_a^b|f|\mathrm{d}v_g \tag{37.27}$$

We leave the proof of this lemma, with $g$ assumed continuous, to homework. Discontinuities of $g$ are handled by a separate argument. The generalized Stieltjes integrability fares better in this context.

### 37.3 Young integral.

The assumption that $f$ is continuous and $g$ is bounded variation, or *vice versa*, is the one most commonly made in the literature on the Stieltjes integral. However, this is not the end of the story; indeed, one can trade regularity of $g$ against regularity of $f$. This was pushed by L.C. Young in the 1930s and reappeared quite usefully in stochastic analysis over the last two decades. The starting point is an inequality that uses the notion of $p$-variation of $f\colon[a,b]\to\mathbb{R}$ defined, for $p>0$, by

$$V^p\big(f,[a,b]\big):=\sup_{n\geqslant1}\;\sup_{\Pi=\{t_i\}_{i=0}^n}\;\sum_{i=1}^n\big|f(t_i)-f(t_{i-1})\big|^p \tag{37.28}$$

We then have:

**Lemma 37.10** (Love-Young inequality) *Let $a<b$ be reals and $f,g\colon[a,b]\to\mathbb{R}$ functions such that $V^p(f,[a,b])<\infty$ and $V^q(g,[a,b])<\infty$. Then for all $p,q>0$, all natural $n\geqslant1$, all marked partition $\Pi=(\{t_i\}_{i=0}^n,\{t_i^\star\}_{i=1}^n)$ of $[a,b]$ into $n$ intervals and all $t\in[a,b]$,*

$$\begin{aligned}&\Big|S(f,\mathrm{d}g,\Pi)-f(t)\big[g(b)-g(a)\big]\Big|\\&\qquad\leqslant\left(1+\sum_{k=1}^{n-1}\frac{1}{k^{1/p+1/q}}\right)V^p\big(f,[a,b]\big)^{1/p}\,V^q\big(g,[a,b]\big)^{1/q},\end{aligned} \tag{37.29}$$

*where the sum over $k$ is treated as zero when $n=1$.*

*Proof.* The proof (drawn vaguely from L.C. Young's paper "An inequality of the Hölder type, connected with Stieltjes integration" in *Acta Mathematica* in 1938) hinges on the following observation: Given a natural $n\geqslant1$ and reals $a_1,\dots,a_n,b_1,\dots,b_n\in[0,\infty)$, let $k = 1, \dots, n$ be such that $a_k b_k = \min_{i=1,\dots,n} a_i b_1$. Then the multivariate AMGM inequality

$$\forall x_1, \dots, x_n \geqslant 0\colon \ \Bigl(\prod_{i=1}^n x_i\Bigr)^{1/n} \leqslant \frac{1}{n}\sum_{i=1}^n x_i \tag{37.30}$$

gives, for each $p, q > 0$, that

$$a_k b_k \leqslant \Bigl(\prod_{i=1}^n a_i^p\Bigr)^{\frac{1}{pn}} \Bigl(\prod_{i=1}^n b_i^q\Bigr)^{\frac{1}{qn}} \leqslant \frac{1}{n^{1/p+1/q}} \Bigl(\sum_{i=1}^n a_i^p\Bigr)^{1/p} \Bigl(\sum_{i=1}^n b_i^q\Bigr)^{1/q} \tag{37.31}$$

As we will see, this opens up the possibility to prove the claim by induction.

Fix $t \in [a, b]$ and reals $p, q > 0$. For the base case $n = 1$ of partition $\Pi$ consisting of just one interval $[a, b]$ and a partition point $t'$, we have

$$S(f, \mathrm{d}g, \Pi) - f(t)\bigl[g(a) - g(b)\bigr] = \bigl[f(t') - f(t)\bigr]\bigl[g(b) - g(a)\bigr] \tag{37.32}$$

Assuming, without simplicity of notation, that $t' > t$, then

$$\begin{aligned} \bigl|f(t') - f(t)\bigr| &= \Bigl(\bigl|f(t') - f(t)\bigr|^p\Bigr)^{1/p} \\ &\leqslant \Bigl(\bigl|f(t) - f(a)\bigr|^p + \bigl|f(t') - f(t)\bigr|^p + \bigl|f(b) - f(t')\bigr|^p\Bigr)^{1/p} \\ &\leqslant V^p\bigl(f, [a, b]\bigr)^{1/p} \end{aligned} \tag{37.33}$$

and using that, trivially, $|g(b) - g(t)| \leqslant V^q\bigl(g, [a, b]\bigr)^{1/q}$ gives

$$\Bigl|S(f, \mathrm{d}g, \Pi) - f(t)\bigl[g(a) - g(b)\bigr]\Bigr| \leqslant V^p\bigl(f, [a, b]\bigr)^{1/p}\, V^q\bigl(g, [a, b]\bigr)^{1/q} \tag{37.34}$$

thus proving the claim for $n = 1$.

Next suppose that the claim holds for a natural $n$ and let $\Pi = (\{t_i\}_{i=0}^{n+1}, \{t_i^\star\}_{i=1}^{n+1})$ be a partition of $[a, b]$ into $n + 1$ intervals. Let $k = 1, \dots, n$ be the smallest index such that

$$\bigl[f(t_{k+1}^\star) - f(t_k^\star)\bigr]\bigl[g(t_k) - g(t_{k-1})\bigr] = \min_{i=1,\dots,n} \bigl[f(t_{i+1}^\star) - f(t_i^\star)\bigr]\bigl[g(t_i) - g(t_{i-1})\bigr] \tag{37.35}$$

Now let $\Pi'$ be a partition obtained by removing partition point $t_i$ and marked point $t_i^\star$ from $\Pi$. The intervals $[t_{i-1}, t_i]$ and $[t_i, t_{i+1}]$ in $\Pi$ are thus united into $[t_{i-1}, t_{i+1}]$ in $\Pi'$ and the latter interval now receives marked point $t_{i+1}^\star$. As all other intervals and marked points remain the same, this gives

$$\begin{aligned} S(f, \mathrm{d}g, \Pi') - S(f, \mathrm{d}g, \Pi) &= f(t_{k+1}^\star)\bigl[g(t_{k+1}) - g(t_{k-1})\bigr] \\ &\qquad - f(t_{k+1}^\star)\bigl[g(t_{k+1}) - g(t_k)\bigr] - f(t_k^\star)\bigl[g(t_k) - g(t_{k-1})\bigr] \\ &= \bigl[f(t_{k+1}^\star) - f(t_k^\star)\bigr]\bigl[g(t_k) - g(t_{k-1})\bigr] \end{aligned} \tag{37.36}$$

The inequality (37.31) enabled by (37.35) then gives

$$\begin{aligned} &\Bigl|S(f, \mathrm{d}g, \Pi') - S(f, \mathrm{d}g, \Pi)\Bigr| \\ &\qquad \leqslant \frac{1}{n^{1/p+1/q}} \Bigl(\sum_{j=1}^n \bigl|f(t_{i+1}^\star) - f(t_i^\star)\bigr|^p\Bigr)^{1/p} \Bigl(\sum_{j=1}^n \bigl|g(t_i) - g(t_{i-1})\bigr|^q\Bigr)^{1/q} \\ &\qquad \leqslant \frac{1}{n^{1/p+1/q}}\, V^p\bigl(f, [a, b]\bigr)^{1/p}\, V^q\bigl(g, [a, b]\bigr)^{1/q} \end{aligned} \tag{37.37}$$

Using that

$$\begin{aligned}\Big|S(f,\mathrm{d}g,\Pi)-f(t)\big[g(b)-g(a)\big]\Big| \\ \leqslant \Big|S(f,\mathrm{d}g,\Pi')-S(f,\mathrm{d}g,\Pi)\Big|+\Big|S(f,\mathrm{d}g,\Pi')-f(t)\big[g(b)-g(a)\big]\Big|\end{aligned} \tag{37.38}$$

the claim for $\Pi$ follows by combining (37.37) with the claim for $\Pi'$, which is TRUE thanks to the induction assumption and the fact that $\Pi'$ partitions $[a,b]$ into only $n$ intervals. □

We now put the above inequality to a good use in:

**Theorem 37.11** (L.C. Young) *Let $a<b$ be reals and $f,g\colon[a,b]\to\mathbb{R}$ functions such that $f$ is $\alpha$-Hölder and $g$ is $\beta$-Hölder for some $\alpha,\beta>0$ with $\alpha+\beta>1$. Then $f\in\mathrm{RS}(g,[a,b])$ and $g\in\mathrm{RS}(f,[a,b])$.*

*Proof.* Let $p>1/\alpha$ and $q>1/\beta$ be such that $\frac{1}{p}+\frac{1}{q}>1$. (This is possible thanks to the assumption $\alpha+\beta>1$.) The assumption that $f$ is $\alpha$-Hölder means that there exists $c>0$ such that $\forall x,y\in[a,b]\colon |f(x)-f(y)|\leqslant c|x-y|^\alpha$. Pick any interval $[s,t]\subseteq[a,b]$ and let $\Pi=\{t_i\}_{i=0}^n$ be a partition of $[s,t]$. Then

$$\begin{aligned}\sum_{i=1}^n\big|f(t_i)-f(t_{i-1})\big|^p &\leqslant c^p\sum_{i=1}^n|t_i-t_{i-1}|^{\alpha p} \\ &\leqslant c^p(t-s)^{\alpha p-1}\sum_{i=1}^n|t_i-t_{i-1}|=c^p(t-s)^{\alpha p}\end{aligned} \tag{37.39}$$

and so

$$V^p\big(f,[s,t]\big)^{1/p}\leqslant c|s-t|^\alpha \tag{37.40}$$

Similarly we get

$$V^q\big(g,[s,t]\big)^{1/q}\leqslant \tilde{c}|t-s|^\beta \tag{37.41}$$

where $\tilde{c}$ is the constant such that $\forall x,y\in[a,b]\colon |g(x)-g(y)|\leqslant\tilde{c}|x-y|^\beta$.

Let $\zeta(s):=\sum_{n=1}^\infty\frac{1}{n^s}$ be the Riemann zeta function where we note the series converges as soon as $s>1$. Given $\epsilon>0$, let $\delta>0$ be such that

$$c\tilde{c}(b-a)\big[1+\zeta(1/p+1/q)\big]\delta^{\alpha+\beta-1}<\epsilon \tag{37.42}$$

and let $\Pi=(\{t_i\}_{i=0}^n,\{t_i^\star\}_{i=1}^n)$ and $\Pi=(\{t_i'\}_{i=0}^m,\{t_i^{\star\star}\}_{i=1}^m)$ be partitions of $[a,b]$ satisfying $\|\Pi\|,\|\Pi'\|<\delta$. Assuming first that $\Pi'$ obeys $\Pi\subseteq\Pi'$, for each $i=1,\dots,m$, let $\Pi_i'$ be the partition of $[t_{i-1},t_i]$ induced by $\Pi'$. Lemma 37.10 now gives

$$\begin{aligned}\Big|S(f,\mathrm{d}g,\Pi_i')-f(t_i^\star)\big[g(t_i)-g(t_{i-1})\big]\Big| \\ \leqslant \big[1+\zeta(1/p+1/q)\big]\,V^p\big(f,[t_{i-1},t_i]\big)^{1/p}\,V^q\big(g,[t_{i-1},t_i]\big)^{1/q} \\ \leqslant c\tilde{c}\big[1+\zeta(1/p+1/q)\big](t_i-t_{i-1})^{\alpha+\beta}<\frac{\epsilon}{b-a}(t_i-t_{i-1})\end{aligned} \tag{37.43}$$

where we also invoked (37.40–37.41) and used $t_i - t_{i-1} < \delta$ along with (37.42). Hereby we get

$$\begin{aligned}\Bigl|S(f,\mathrm{d}g,\Pi') - S(f,\mathrm{d}g,\Pi)\Bigr| &\leqslant \sum_{i=1}^n \Bigl|S(f,\mathrm{d}g,\Pi_i') - f(t_i^\star)\bigl[g(t_i)-g(t_{i-1})\Bigr| \\ &\leqslant \sum_{i=1}^n \frac{\epsilon}{b-a}(t_i - t_{i-1}) = \epsilon\end{aligned} \tag{37.44}$$

When $\Pi'$ is not a refinement of $\Pi$, then by going to their common refinement we bound the difference instead by $2\epsilon$. As $\epsilon > 0$ was arbitrary, the Cauchy criterion (cf Lemma 37.6) now implies $f \in \mathrm{RS}(g,[a,b])$ and, by Lemma 37.6, also $g \in \mathrm{RS}(f,[a,b])$. □

A slightly more sophisticated argument shows that the finiteness of the $p$-variation of $f$ and $q$-variation of $g$ for some $p,q > 0$ with $\frac{1}{p} + \frac{1}{q} > 1$ are sufficient for integrability of $f$ with respect to $g$ and *vice versa*. Young also showed that assuming this for $p,q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$ is not enough. The Stieltjes integral derived under these conditions is sometimes referred to as the *Young integral* even though what Young's work does is to provide a useful sufficient condition for Stieltjes integrability rather than defining a new integration theory in its own right.