36. STIELTJES INTEGRAL

The Riemann integral admits a natural generalization that was invented by T.J. Stieltjes 1890s in his work on continued fractions. This generalization is useful in applications and also served as a foundation for the corresponding extension of the Lebesgue integration theory. (Stieljes had an interesting career path to mathematics. He died in 1894 at the age of 38, too early to see his work published and his integral gain prominence.)

36.1 Stieltjes integral: definition and motivation.

The main distinction of the Stietjes integral $\int_a^b f \mathrm{d}g$ from the Riemann integral is that it depends on two functions: First, the integrand f and then a function g that replaces the identity map (i.e., $g(x) = x$) in the increment of the underlying variable. Precise definitions are as follows:

Definition 36.1 (Stieltjes integral) *Let $a < b$ be reals and $f, g\colon [a,b] \to \mathbb{R}$ functions. Given a marked partition $\Pi = (\{t_i\}_{i=0}^n, \{t_i^\star\}_{i=1}^n)$ — i.e., two sequences of reals subject to the requirements (31.1–31.2) — we define the* Riemann-Stieltjes sum *by*

$$S(f, \mathrm{d}g, \Pi) := \sum_{i=1}^n f(t_i^\star)\big(g(t_i) - g(t_{i-1})\big) \tag{36.1}$$

The function f is then said to be Stieltjes integrable with respect to g on $[a,b]$ in Riemann sense *(or sometimes called* Riemann-Stieltjes integrable*) if*

$$\int_a^b f \mathrm{d}g := \lim_{\|\Pi\|\to 0} S(f, \mathrm{d}g, \Pi) \text{ exists} \tag{36.2}$$

where the "limit" abbreviates the same concept as for the Riemann integral (see Definition 31.2). We call the object on the left the Stieltjes integral of f with respect to g.

Some remarks are in order. The above is often referred to as the *Riemann-Stieltjes integral*. This is not because Riemann had anything to do with it but rather that the integral uses the framework of the Riemann integral. (A version called *Lebesgue-Stieltjes integral* exists in Lebesgue theory of integration.) The Riemann integral is thus a special case of the (Riemann-)Stieltjes integral; indeed,

$$\big(\forall x \in [a,b]\colon\ g(x) = x\big) \ \Rightarrow\ \int_a^b f \mathrm{d}g = \int_a^b f(x)\mathrm{d}x \tag{36.3}$$

Turning this around, the Stieltjes integral allows us to generalize the notion of area to the situation when the "length" of interval $[s,t]$ is given $g(t) - g(s)$ and area of the rectangle $[s,t] \times [0,h]$ is thus $h[g(t) - g(s)]$. Notably, this includes negative "lengths" but this is no problem because the Riemann integral anyway computes the signed area.

The Stieltjes integral is quite useful in probability. There g is usually the *cumulative distribution function* of a random variable X; meaning that

$$g(x) := P(X \leqslant x) \tag{36.4}$$

The integral $\int_a^b f \mathrm{d}g$ then corresponds to the *expectation* of $f(X)$; i.e., the statistical "mean value" of the random variable obtained by plugging X into f. The use of the Stieltjes integral permits treating all the various kinds of distributions of X — namely, discrete, continuous and mixtures thereof — under the same umbrella.

The fact that g need not be monotone, and the increments $g(t) - g(s)$ over interval $[s,t]$ thus need not be positive, is quite advantageous in applications. For instance, consider a position in stock portfolio whose volume at time t is described by $g(t)$. The quantity

$$f(t_i^\star)\big[g(t_i) - g(t_{i-1})\big] \tag{36.5}$$

is then the price paid for (if positive) or earned from (if negative) the change in portfolio over time interval $[t_{i-1}, t_i]$ assuming all trade was executed at the instanteous price at time $t_i^\star \in [t_{i-1}, t_i]$. The Stietljes sum thus approximates the total cash value traded over the time interval $[a,b]$ which, in the limit as the as mesh of the partition tends to zero, is thus given by the integral $\int_a^b f\mathrm{d}g$.

We note that Rudin's book presents a different definition of the Stieltjes integral which is based on Darboux's approach to Riemann integration. This streamlines the analysis somewhat but forces us to work with g monotone or, at best, of bounded variation. The applications mentioned earlier are not always of this kind and so we prefer to work with the Stieltjes integral in the Riemann sense.

36.2 Properties of Stieltjes integral.

We now move to discuss the basic properties of Stieltjes integral. Let us henceforth use

$$\mathrm{RS}\big(h,[a,b]\big) := \Big\{f\colon [a,b] \to \mathbb{R} : \int_a^b f\mathrm{d}h \text{ exists}\Big\} \tag{36.6}$$

to denote the set of functions that are Stieltjes-integrable with respect to g on $[a,b]$ in Riemann sense. We start with some "good news;" namely, facts where the Stieltjes integral behaves very much as Riemann's:

Lemma 36.2 (Linearity) *Let $a < b$ be reals and let $h\colon [a,b] \to \mathbb{R}$ be given. Then for all $f,g \in \mathrm{RS}(h,[a,b])$ and all $\alpha,\beta \in \mathbb{R}$,*

$$\alpha f + \beta g \in \mathrm{RS}\big(h,[a,b]\big) \tag{36.7}$$

and

$$\int_a^b \big(\alpha f + \beta g\big)\mathrm{d}h = \alpha\int_a^b f\mathrm{d}h + \beta\int_a^b g\mathrm{d}h \tag{36.8}$$

Proof. The proof is the exactly the same as for the Riemann integral (see Lemma 31.5) and so we leave it to the reader. □

A similar type of linearity holds also for the integrator:

Lemma 36.3 *For all $\alpha,\beta \in \mathbb{R}$, all $g_1, g_2\colon [a,b] \to \mathbb{R}$ and all $f \in \mathrm{RS}(g_1,[a,b]) \cap \mathrm{RS}(g_2,[a,b])$,*

$$f \in \mathrm{RS}\big(\alpha g_1 + \beta g_2, [a,b]\big) \;\wedge\; \int_a^b f\mathrm{d}(\alpha g_1 + \beta g_2) = \alpha\int_a^b f\mathrm{d}g_1 + \beta\int_a^b f\mathrm{d}g_2 \tag{36.9}$$

Proof. Left to the reader. □

Moving to properties where the Stietljes integral behaves somewhat differently than (its special case of) the Riemann integral we note that the boundedness that came with

Riemann integrability is no longer applicable. Indeed, on intervals $[s,t] \subseteq [a,b]$ where g is constant and the values of $\{f(x)\colon x \in (s,t)\}$ do not contribute to $S(f, \mathrm{d}g, \Pi)$ for any partition Π and are thus completely unconstrained by the assumption that f is integrable with respect to g. Turning to the positive side of the story, we get:

Lemma 36.4 *Let $f, g\colon [a,b] \to \mathbb{R}$ be such that $f \in \mathrm{RS}(g,[a,b])$. Then there exists a partition $\Pi = \{t_i\}_{i=0}^n$ of $[a,b]$ such that*

$$\forall i = 1,\dots,n\colon \quad \sup_{x\in[t_{i-1},t_i]} \bigl|f(x)\bigr| < \infty \;\vee\; \bigl(\forall x,y \in [t_{i-1},t_i]\colon\; g(x) = g(y)\bigr) \tag{36.10}$$

In particular, if g is NOT constant on any non-degenerate closed subinterval of $[a,b]$, then $f \in \mathrm{RS}(g,[a,b])$ implies that f is bounded.

As it turns out, also the additivity property with respect to the underlying domain holds only in a restricted sense. Namely, we have:

Lemma 36.5 *For all reals $a < c < b$ and all $f, g\colon [a,b] \to \mathbb{R}$,*

$$f \in \mathrm{RS}(g,[a,b]) \;\Rightarrow\; f \in \mathrm{RS}(g,[a,c]) \wedge f \in \mathrm{RS}(g,[c,b]) \tag{36.11}$$

and, in particular,

$$\forall f \in \mathrm{RS}(g,[a,b])\colon \quad \int_a^b f\mathrm{d}g = \int_a^c f\mathrm{d}g + \int_c^b f\mathrm{d}g \tag{36.12}$$

Proof. Left to the reader. □

The previous lemma worked because it assumed integrability on the larger domain. Unlike the Riemann integral, for which (36.11) is an equivalence, for the Stieltjes integral one can have integrability on $[a,c]$ and on $[c,b]$ without having integrability on $[a,b]$. This is because of the following necessary condition for integrability:

Lemma 36.6 *Let $f, g\colon [a,b] \to \mathbb{R}$. If $f \in \mathrm{RS}(g,[a,b])$ then for each $\epsilon > 0$ there is $\delta > 0$ such that for any unmarked partition $\Pi = (\{t_i\}_{i=0}^n)$ of $[a,b]$,*

$$\|\Pi\| < \delta \;\Rightarrow\; \sum_{i=1}^n \mathrm{osc}\bigl(f,[t_{i-1},t_i]\bigr)\bigl|g(t_i) - g(t_{i-1})\bigr| < \epsilon \tag{36.13}$$

In particular, if $f \in \mathrm{RS}(g,[a,b])$, then f and g have no common discontinuity points.

Proof. To get (36.13), consider two Stieltjes sums for partitions $\Pi^\star$ and $\Pi^{\star\star}$ with the same partition points $\{t_i\}_{i=0}^n$ but the marked points $t_i^\star, t_i^{\star\star} \in [t_{i-1},t_i]$ chosen such that, for each $i = 1,\dots,n$, the difference $f(t_i^\star) - f(t_i^{\star\star})$ has the same sign as $g(t_i) - g(t_{i-1})$ and has absolute value at least $\frac12\mathrm{osc}(f,[t_{i-1},t_i])$. Then

$$\frac{1}{2}\sum_{i=1}^n \mathrm{osc}\bigl(f,[t_{i-1},t_i]\bigr)\bigl|g(t_i) - g(t_{i-1})\bigr| \leqslant S(f,\mathrm{d}g,\Pi^\star) - S(f,\mathrm{d}g,\Pi^{\star\star}) \tag{36.14}$$

Assuming Stieltjes integrability $f \in \mathrm{RS}(g,[a,b])$ and δ related to δ as in Definition 31.2, the right-hand side is smaller than 2ϵ once $\|\Pi^\star\| = \|\Pi^{\star\star}\| < \delta$.

For the second part note that suppose first that f fails to be right-continuous at some $x \in [a,b)$. Using the same ideas as in Lemma 34.5(2), this implies

$$c := \inf_{\delta \in (0,b-x)} \operatorname{osc}(f,[x,x+\delta]) > 0 \tag{36.15}$$

Letting $\delta > 0$ be related to $\epsilon > 0$ as in (36.13), for partitions where x is among the partition points we then get

$$\forall t \in (x,x+\delta) \cap [a,b]\colon \quad \bigl|g(t)-g(x)\bigr| < \frac{1}{c}\epsilon \tag{36.16}$$

As this holds for all $\epsilon > 0$, we conclude that g is right-continuous at x. A similar proof gives left-continuity of g at all points $x \in (a,b]$ where f is NOT left continuous.

It remains to rule out the possibility that f is NOT right-continuous and g is NOT left-continuous but is right continuous at some $x \in (a,b)$. Here we take a partition that does NOT contain x among its partition points. Then we still have

$$\tilde{c} := \inf_{a \leqslant s < x < t \leqslant b} \operatorname{osc}\bigl(f,[s,t]\bigr) > 0 \tag{36.17}$$

and so (36.13) gives (for δ related to ϵ as above),

$$\forall s,t \in [a,b]\colon \; s < x < t \,\wedge\, t-s < \delta \,\Rightarrow\, \bigl|g(t)-g(s)\bigr| < \frac{1}{\tilde{c}}\epsilon \tag{36.18}$$

If g is assumed right continuous, then taking $t \to x^+$ shows that $|g(x)-g(s)| \leqslant \tilde{c}^{-1}\epsilon$ whenever $0 < x-s < \delta$. But then g is left-continuous as well. (The complementary set of type of dicontinuities is handled similarly.) ☐

The previous proof now explains why $\Leftarrow$ generally fails in (36.11): If $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,c]$ and $(c,b]$ but NOT continuous at c while $g\colon [a,b] \to \mathbb{R}$ is continuous on $[c,b]$ and $[a,c)$ but NOT continuous at c, then

$$f \notin \operatorname{RS}(g,[a,b]) \;\wedge\; f \in \operatorname{RS}(g,[a,c]) \cap \operatorname{RS}(g,[c,b]) \tag{36.19}$$

This is actually a somewhat annoying feature of the Stieltjes integral which stems from the fact that, for g with jumps, we may not be able to refine partition intervals so that the increment over each of them is small. This is partially fixed in:

Definition 36.7 (Generalized Stieltjes integrability) *We say that $f\colon [a,b] \to \mathbb{R}$ is generalized Stieltjes integrable with respect to $g\colon [a,b] \to \mathbb{R}$ if there is $L \in \mathbb{R}$ for each $\epsilon > 0$ there is $\delta > 0$ and an (unmarked) partition Π_ϵ such that for all partitions Π*

$$\Pi_\epsilon \subseteq \Pi \,\wedge\, \|\Pi\| < \delta \,\Rightarrow\, \bigl|S(f,\mathrm{d}g,\Pi) - L\bigr| < \epsilon \tag{36.20}$$

Here $\Pi_\epsilon \subseteq \Pi$ means that the partition points of Π include all the partition points of Π_ϵ.

Note that we can restrict Π_ϵ and δ to $\|\Pi_\epsilon\| < \delta$, which then forces $\|\Pi\| < \delta$ as soon as $\Pi_\epsilon \subseteq \Pi$. The reference to δ is thus redundant and we can leave it out altogether. The resulting integral is called the *Moore-Pollard-Stieltjes integral* by some authors. While, with these generalizations, the integral becomes additive in the underlying domain, this is not such an important improvement and so we will stick to the definition used previously because it makes the proofs somewhat easier.